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CLASSIFICATION OF FINITE GROUPS ACCORDING TO THE NUMBER OF CONJUGACY CLASSES II

BY

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ABSTRACT

In the following, G denotes a finite group, $r(G)$ the number of conjugacy classes of G, $\beta(G)$ the number of minimal normal subgroups of G and $\alpha(G)$ the number of conjugate classes of G not contained in the socle $S(G)$. Let $\Phi_j = \{G \mid \beta(G) = r(G) - j\}$. In this paper, the family Φ_{11} is classified. In addition, from a simple inspection of the groups with $r(G) = b$ conjugate classes that appear in $\dot{\bigcup}_{j=1}^{11} \Phi_j$, we obtain all finite groups satisfying one of the following conditions: (1) $r(G) = 12$; (2) $r(G) = 13$ and $\beta(G) > 1$; ...; (9) $r(G) = 20$ and $\beta(G) > 8$; (10) $r(G) = n$ and $\beta(G) = n - a$ with $1 \leqq a \leqq 11$, for each integer $n \geqq 21$. Also, we obtain all finite groups G with $13 \leqq r(G) \leqq 20$, $\beta(G) \leqq r(G) - 12$, and satisfying one of the following conditions: (i) $0 \leqq \alpha(G) \leqq 4$; (ii) $5 \leqq \alpha(G) \leqq 10$ and $S(G)$ solvable.

1. Introduction

In this work, G will denote a finite group, $r = r(G)$ the number of conjugacy classes, $\beta(G)$ the number of minimal normal subgroups of G, and $\alpha(G)$ the number of conjugate classes of G not contained in the socle $S(G)$.

The possibility of classifying finite groups according to the number $r(G)$ and to some properties of their conjugacy classes was suggested in [2].

The classification of all finite groups with $r(G) \leqq 9$ was carried out in a series of papers by G. A. Miller and W. Burnside ($r(G) \leqq 5$, cf. [2] Note A, 1910), D. I. Sigley ($r(G) = 6$, [21], 1935), J. Poland ($r(G) = 7$, [19], 1966), L. F. Kosvintsev ($r(G) = 8$, [12], 1974) and V. A. Odincov, A. I. Starostin ($r(G) = 9$, [17], 1976). In 1978, A. G. Aleksandrov and K. A. Komissarcik ([1]) found all finite simple groups with $r(G) \leqq 12$.

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In [25] we approached the problem of classifying finite groups according to the number $r(G)$ through the classification of the families $\Phi_j = \{G \mid \beta(G) = r(G) - j\}$ for small values of the natural number j. The families Φ_i, $i = 1, 2, \ldots, 10$ are classified and as an immediate corollary, the previously known classification of finite groups with $r(G) \leqq 9$ is found, as well as that of those finite groups satisfying one of the following conditions:

(i) $r(G) = 10$,

(ii) $r(G) = 11$,

(iii) $r(G) = n$ and $\beta(G) = n - a$ with $1 \leqq a \leqq 10$, for each integer $n \geqq 12$.

In this paper, all groups G with $\beta(G) = r(G) - 11$ are classified. Using the results of [25], we obtain as an immediate corollary all finite groups satisfying one of the following conditions:

(1) $r(G) = 12$,

(2) $r(G) = 13$ and $\beta(G) > 1$,

(3) $r(G) = 14$ and $\beta(G) > 2$,

. . . .

(9) $r(G) = 20$ and $\beta(G) > 8$,

(10) $r(G) = n$ and $\beta(G) = n - a$ with $1 \leqq a \leqq 11$, for each integer $n \geqq 21$.

Moreover, we obtain all finite groups G with $13 \leqq r(G) \leqq 20$ and $\beta(G) \leqq r(G) - 12$, and satisfying one of the following conditions:

(a) $0 \leqq \alpha(G) \leqq 4$,

(b) $5 \leqq \alpha(G) \leqq 10$ and $S(G)$ solvable.

We shall follow closely the notation introduced in [25]. If $\varnothing \neq S \subseteq G$, we define

$$r_G(S) = |\{\mathrm{Cl}_G(g) \mid \mathrm{Cl}_G(g) \cap S \neq \varnothing\}|.$$

In addition, if S is a normal set in G, we define $\Delta_S^G = (|C_G(x_1)|, \ldots, |C_G(x_t)|)$, if $|C_G(x_1)| \geqq \cdots \geqq |C_G(x_t)|$ and $S = \mathrm{Cl}_G(x_1) \dot\cup \cdots \dot\cup \mathrm{Cl}_G(x_t)$. In particular, if $S(G)$ denotes the socle of G and $S_0 = \bigcup_{g \in G} (xS(G))^g$, then we write $\Delta_x = \Delta_x^G = \Delta_{S_0}^G$. Finally, in case $S = G$, we set $\Delta_S^G = \Delta_G$.

Also, $\langle a \rangle = C_m$ denotes a cyclic group of order m generated by a, $\hat{\Sigma}_m^{(i)}$ denotes the two non-isomorphic proper coverings of Σ_m by C_2, and

$$M_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1 = b^2, a^b = a^{1+2^{n-1}} \rangle$$

denotes the ordinary non-abelian group of order 2^n.

Now, the finite groups satisfying conditions (1)–(9) are described in Tables 1–9. These tables list the r-tuples Δ_G and the structures of $G/S(G)$.

TABLE 1
The finite groups with exactly twelve conjugacy classes

G	Δ_G	$G/S(G)$	Reference
$C_2 \times C_6$	$(12,\overset{12}{\ldots\ldots},12)$	1	(2.17) [25]
C_{12}	$(12,\overset{12}{\ldots\ldots},12)$	C_2	(2.17) [25]
$C_3 \times_\lambda C_8$	$(24,\overset{4}{\ldots},24,12,\overset{4}{\ldots},12,8,\overset{4}{\ldots},8)$	C_4	(4.2) [25]
$C_4 \times \Sigma_3$	$(24,\overset{4}{\ldots},24,12,\overset{4}{\ldots},12,8,\overset{4}{\ldots},8)$	C_2^2	(4.2) [25]
$C_2^2 \times \Sigma_3$	$(24,\overset{4}{\ldots},24,12,\overset{4}{\ldots},12,8,\overset{4}{\ldots},8)$	C_2	(4.1) [25]
$C_2 \times \mathrm{DC}_3$	$(24,\overset{4}{\ldots},24,12,\overset{4}{\ldots},12,8,\overset{4}{\ldots},8)$	C_2	(4.1) [25]
$C_3 \times D_{10}$	$(30,30,30,15,\overset{6}{\ldots},15,6,6,6)$	C_2	(2.20) [25]
$C_2 \times (C_9 \times_f C_2)$	$(36,36,18,\overset{8}{\ldots},18,4,4)$	Σ_3	(4.2) [25]
$C_9 \times_\lambda C_4$	$(36,36,18,\overset{8}{\ldots},18,4,4)$	Σ_3	(4.2) [25]
$C_3 \times A_4$	$(36,36,36,12,12,12,9,\overset{6}{\ldots},9)$	C_3	(4.2) [25]
$C_3^2 \times_\lambda C_4$	$(36,36,18,\overset{8}{\ldots},18,4,4)$	C_2	(2.19) [25]
$C_2^2 \times_\lambda C_9$	$(36,36,36,12,12,12,9,\overset{6}{\ldots},9)$	C_3	(4.2) [25]
$(C_3 \times C_7) \times_f C_2$	$(42,21,\overset{10}{\ldots\ldots},21,2)$	C_2	(2.18) [25]
$C_3 \times_\lambda Q_{16}$	$(48,48,24,24,24,12,\overset{4}{\ldots},12,8,8,4)$	D_8	(4.2) [25]
$C_3 \times_\lambda D_{16}$	$(48,48,48,24,24,12,\overset{4}{\ldots},12,8,8,4)$	D_8	(4.2) [25]
$C_3 \times_\lambda \mathrm{SD}_{16}$	$(48,48,48,24,24,12,\overset{4}{\ldots},12,8,8,4)$	D_8	(4.2) [25]
$\Sigma_3 \times D_{10}$	$(60,30,30,30,20,15,15,12,10,10,6,4)$	C_2^2	(4.2) [25]
$C_3^2 \times_\lambda C_8$	$(72,72,18,\overset{4}{\ldots},18,8,\overset{6}{\ldots},8)$	C_4	(4.2) [25]
$\Sigma_3 \times A_4$	$(72,36,24,24,18,18,12,9,9,8,6,6)$	C_6	(4.2) [25]
$C_2 \times (C_3^2 \times_f C_4)$	$(72,72,18,\overset{4}{\ldots},18,8,\overset{6}{\ldots},8)$	C_4	(4.2) [25]
$(C_2^2 \times C_7) \times_f C_3$	$(84,28,\overset{9}{\ldots},28,3,3)$	C_3	(2.19) [25]
$(C_2^2 \times Q_8) \times_\lambda C_3$	$(96,96,32,32,16,\overset{4}{\ldots},16,6,\overset{4}{\ldots},6)$	A_4	(4.2) [25]

TABLE 1 (contd.)

G	Δ_G	$G/S(G)$	Reference
$\mathrm{Hol}(2^5\Gamma_2h, C_3)$	$(96,96,32,32,16,\overset{4}{\ldots},16,6,\overset{4}{\ldots},6)$	A_4	(4.2) [25]
$C_3^3\times_\lambda C_4$	$(108,54,27,\overset{6}{\ldots},27,6,4,4)$	C_4	(4.1) [25]
$(C_3\times C_7)\times_\lambda C_6$	$(126,63,21,21,21,18,18,9,9,6,6,6)$	C_6	(4.2) [25]
$C_2\times(C_3^2\times_f Q_8)$	$(144,144,18,18,16,16,8,\overset{6}{\ldots},8)$	Q_8	(4.2) [25]
$C_3^2\times_\lambda(C_4\times_\lambda C_4)$	$(144,144,18,18,16,16,8,\overset{6}{\ldots},8)$	Q_8	(4.2) [25]
$(C_2^2\times C_7)\times_\lambda C_6$	$(168,56,28,\overset{4}{\ldots},28,24,8,6,\overset{4}{\ldots},6)$	C_6	(4.2) [25]
$C_3^3\times_\lambda Q_8$	$(216,108,27,27,27,24,12,\overset{4}{\ldots},12,4,4)$	Q_8	(4.2) [25]
$C_{37}\times_f C_6$	$(222,37,\overset{6}{\ldots},37,6,\overset{5}{\ldots},6)$	C_6	(4.2) [25]
$\Sigma_5^{(1)}$	$(240,240,12,\overset{5}{\ldots},12,10,10,8,8,8)$	Σ_5	(1.14)
$\Sigma_5^{(2)}$	$(240,240,12,\overset{5}{\ldots},12,10,10,8,8,8)$	Σ_5	(1.14)
$\mathrm{PSL}(2,7)\times C_2$	$(336,336,16,16,14,\overset{4}{\ldots},14,8,8,6,6)$	$\{1\}$	(3.2) [25]
$(A_5\times C_3)\times_\lambda C_2$	$(360,180,24,18,15,15,15,12,12,9,6,4)$	C_2	(2.20) [25]
$C_2^4\times_\lambda \Sigma_4$	$(384,128,32,32,32,16,16,16,8,8,8,3)$	$C_2^4\times_\lambda \Sigma_3$	(1.14)
$P_1\times_\lambda C_6$	$(384,128,32,32,32,16,16,16,8,8,8,3)$	$C_2^4\times_\lambda C_6$	(1.14)
$P_2\times_\lambda C_6$	$(384,128,32,24,16,16,16,16,6,6,6,6)$	$C_2^4\times_\lambda C_6$	(1.14)
$C_2^4\times_{\lambda_2} A_5$	$(960,192,96,16,16,12,\overset{4}{\ldots},12,8,5,5)$	A_5	(4.2) [25]
$C_3^4\times_f Q_{16}$	$(1296,81,\overset{5}{\ldots},81,16,8,8,8,4,4)$	Q_{16}	(4.5) [25]
$C_3^4\times_f(C_5\times_\lambda C_4)$	$(1620,81,\overset{4}{\ldots},81,20,10,\overset{4}{\ldots},10,4,4)$	$C_5\times_\lambda C_4$	(4.8) [25]
$C_2^4\times_\lambda \Sigma_5$	$(1920,128,48,32,32,16,16,8,8,6,6,5)$	Σ_5	(1.14)
$C_{11}^2\times_f \mathrm{SL}(2,3)$	$(2.904,121,\overset{5}{\ldots},121,24,6,\overset{4}{\ldots},6,4)$	$\mathrm{SL}(2,3)$	(4.5) [25]
$C_3^4\times_f(C_5\times_\lambda C_8)$	$(3240,81,81,40,10,10,8,\overset{6}{\ldots},8)$	$C_5\times_\lambda C_8$	(4.14) [25]
$\mathrm{PSL}(2,19)$	$(3420,20,19,19,10,\overset{4}{\ldots},10,9,\overset{4}{\ldots},9)$	$\{1\}$	(1.13)
$\mathrm{PSL}(3,3)$	$(5616,54,48,13,\overset{4}{\ldots},13,9,8,8,8,6)$	$\{1\}$	(1.13)
$C_2^4\times_\lambda A_6$	$(5760,384,36,32,32,16,12,9,8,8,5,5)$	A_6	(1.14)
$C_{19}^2\times_f \mathrm{SL}(2,5)$	$(43320,361,\overset{3}{\ldots},361,120,10,\overset{4}{\ldots},10,6,6,4)$	$\mathrm{SL}(2,5)$	(4.11) [25]
M_{22}	$(443520,384,36,32,16,12,11,11,8,7,7,5)$	$\{1\}$	(1.13)

TABLE 2

(i) The finite groups satisfying $r(G) = 13$ and $\beta(G) > 1$

G	Δ_G	$G/S(G)$	Reference
$C_5 \times_{\lambda_1} D_8$	$(40, 40, 20, \overset{9}{\dots}, 20, 4, 4)$	C_2^2	(4.2) [25]
$C_5 \times_{\lambda_2} D_8$	$(40, 40, 20, \overset{9}{\dots}, 20, 4, 4)$	C_2^2	(4.2) [25]
$C_5 \times_{\lambda} Q_8$	$(40, 40, 20, \overset{9}{\dots}, 20, 4, 4)$	C_2^2	(4.2) [25]
$C_5^2 \times_{\lambda} C_4$	$(100, 50, 50, 25, \overset{5}{\dots}, 25, 20, 10, 10, 4, 4)$	C_4	(4.2) [25]
$C_2^2 \times_{\lambda} (C_{15} \times_f C_2)$	$(120, 60, 60, 40, 20, 20, 15, \overset{5}{\dots}, 15, 4, 4)$	Σ_3	(4.2) [25]

(ii) The finite groups satisfying $r(G) = 13$, $\beta(G) = 1$ and $0 \leqq \alpha(G) \leqq 4$

G	Δ_G	$G/S(G)$	Reference
C_{13}	$(13, \overset{13}{\dots\dots}, 13)$	1	(1.16)
$C_{23} \times_f C_2$	$(46, 23, \overset{11}{\dots\dots}, 23, 2)$	C_2	(2.18) [25]
$C_{31} \times_f C_3$	$(93, 31, \overset{10}{\dots}, 31, 3, 3)$	C_3	(2.19) [25]
$C_{37} \times_f C_4$	$(148, 37, \overset{8}{\dots}, 37, 4, 4, 4)$	C_4	(2.20) [25]
$C_{41} \times_f C_5$	$(205, 41, \overset{8}{\dots}, 41, 5, \overset{3}{\dots}, 5)$	C_5	(4.1) [25]

(iii) The finite groups satisfying $r(G) = 13$, $\beta(G) = 1$, $5 \leqq \alpha(G) \leqq 10$ and $S(G)$ solvable

G	Δ_G	$G/S(G)$	Reference
$C_5^2 \times_{\lambda} \Sigma_3$	$(150, 50, \overset{4}{\dots}, 50, 25, 25, 10, \overset{5}{\dots}, 10, 3)$	Σ_3	(4.2) [25]
$C_{23} \times_f C_{11}$	$(253, 23, 23, 11, \dots\dots, 11)$	C_{11}	(1.15)
$C_{43} \times_f C_6$	$(258, 43, \overset{7}{\dots}, 43, 6, \overset{5}{\dots}, 6)$	C_6	(4.2) [25]
$C_{43} \times_f C_7$	$(301, 43, \overset{6}{\dots}, 43, 7, \overset{6}{\dots}, 7)$	C_7	(4.5) [25]
$C_{31} \times_f C_{10}$	$(310, 31, \overset{3}{\dots}, 31, 10, \overset{9}{\dots}, 10)$	C_{10}	(4.14) [25]
$C_5^2 \times_{\lambda} M_{16}$	$(400, 50, 40, 25, 16, 16, 16, 10, 8, \overset{5}{\dots}, 8)$	M_{16}	(1.14)
$C_3^3 \times_{\lambda} A_4$	$(324, 81, 81, 54, 27, 12, 9, \overset{6}{\dots}, 9, 6)$	A_4	(4.2) [25]
$C_{41} \times_f C_8$	$(328, 41, \overset{5}{\dots}, 41, 8, \overset{7}{\dots}, 8)$	C_8	(4.8) [25]
$C_{37} \times_f C_9$	$(333, 37, \overset{4}{\dots}, 37, 9, \overset{8}{\dots}, 9)$	C_9	(4.11) [25]
$C_5^2 \times_{\lambda} M_{16}$	$(400, 50, 40, 25, 16, 16, 16, 10, 8, \overset{5}{\dots}, 8)$	M_{16}	(1.14)
$C_2^4 \times_{\lambda} (C_3^2 \times_f C_4)$	$(576, 96, 64, 36, 16, 16, 12, 9, 8, \overset{5}{\dots}, 8)$	$C_3^2 \times_f C_4$	(1.14) [25]
$C_3^3 \times_{\lambda} (C_{13} \times_f C_3)$	$(1053, 81, 81, 13, \overset{4}{\dots}, 13, 9, \overset{6}{\dots}, 9)$	$C_{13} \times_f C_3$	(1.14)
$C_{11}^2 \times_f (C_5 \times_{\lambda} C_8)$	$(4840, 121, \overset{3}{\dots}, 121, 40, 10, 10, 8, \overset{6}{\dots}, 8)$	$C_5 \times_{\lambda} C_8$	(4.14) [25]

TABLE 3

(i) The finite groups satisfying $r(G)=14$ and $\beta(G)>2$

G	Δ_G	$G/S(G)$	Reference
$C_2\times D_{16}$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	D_8	(1.16)
$C_2\times \mathrm{SD}_{16}$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	D_8	(1.16)
$C_2\times Q_{16}$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	D_8	(1.16)
$(C_8\times C_2)\times_\lambda C_2$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	D_8	(1.16)
$(C_8\times C_2)\cdot C_4$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	D_8	(1.16)
$C_8\times_{\lambda_1} C_4$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	D_8	(1.16)
$C_8\times_{\lambda_2} C_4$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	D_8	(1.16)
$C_2^4\times_\lambda C_2$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	C_2^3	(1.16)
$C_4^2\times_{\lambda_1} C_2$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	C_2^3	(1.16)
$C_4^2\times_{\lambda_2} C_2$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	C_2^3	(1.16)
$C_4^2\underset{1}{\cdot} C_4$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	C_2^3	(1.16)
$C_4^2\underset{2}{\cdot} C_4$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	C_2^3	(1.16)
$(C_4\times C_2^2)\times_{\lambda_1} C_2$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	C_2^3	(1.16)
$(C_4\times C_2^2)\times_{\lambda_2} C_2$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	C_2^3	(1.16)
$(C_4\times C_2^2)\cdot C_4$	$(32,\overset{4}{\ldots},32,16,\overset{6}{\ldots},16,8,\overset{4}{\ldots},8)$	C_2^3	(1.16)
$C_2\times \mathrm{SL}(2,3)$	$(48,\overset{4}{\ldots},48,12,\overset{8}{\ldots},12,8,8)$	A_4	(1.14)
$C_5^2\times_f C_2$	$(50,25,\overset{12}{\ldots\ldots},25,2)$	C_2	(2.18) [25]
$C_7^2\times_{f_2} C_6$	$(294,49,\overset{8}{\ldots},49,6,\overset{5}{\ldots},6)$	C_6	(4.2) [25]

(ii) The finite groups satisfying $r(G)=14$, $\beta(G)\leqq 2$ and $0\leqq\alpha(G)\leqq 4$

G	Δ_G	$G/S(G)$	Reference
C_{14}	$(14,\overset{14}{\ldots\ldots},14)$	1	(1.16)
$C_{11}\times_\lambda C_4$	$(44,44,22,\overset{10}{\ldots\ldots},22,4,4)$	C_2	(2.19) [25]
$C_2\times(C_{11}\times_f C_2)$	$(44,44,22,\overset{10}{\ldots\ldots},22,4,4)$	C_2	(2.19) [25]
$C_2\times(C_{13}\times_f C_3)$	$(78,78,26,\overset{8}{\ldots},26,6,\overset{4}{\ldots},6)$	C_3	(4.1) [25]
$C_{41}\times_f C_4$	$(164,41,\overset{10}{\ldots\ldots},41,4,4,4)$	C_4	(2.20) [25]

TABLE 3 (contd.)

(iii) The finite groups satisfying $r(G)=14$, $\beta(G)\leqq 2$, $5\leqq\alpha(G)\leqq 10$ and $S(G)$ solvable

G	Δ_G	$G/S(G)$	Reference
$C_5\times_{\lambda_1}M_{16}$	$(80,80,40,40,20,20,20,16,16,8,\overset{5}{\ldots},8)$	$C_4\times C_2$	(1.14)
$C_5\times_{\lambda_2}M_{16}$	$(80,80,40,40,20,20,20,16,16,8,\overset{5}{\ldots},8)$	$C_4\times C_2$	(1.14)
$C_5\times_\lambda(C_4\times_\lambda C_4)$	$(80,80,40,20,\overset{4}{\ldots},20,16,16,8,\overset{5}{\ldots},8)$	$C_4\times C_2$	(1.14)
$C_5\times_\lambda((C_4\times C_2)\times_\lambda C_2)$	$(80,80,40,20,\overset{4}{\ldots},20,16,16,8,\overset{5}{\ldots},8)$	$C_4\times C_2$	(1.14)
$C_2\times(C_7\times_f C_6)$	$(84,84,14,14,12,\overset{10}{\ldots\ldots},12)$	C_6	(1.14)
$C_7\times_\lambda C_{12}$	$(84,84,14,14,12,\overset{10}{\ldots\ldots},12)$	C_6	(1.14)
$C_2^2\times_{\lambda_1}(C_3\times_\lambda D_8)$	$(96,96,48,32,32,16,12,\overset{4}{\ldots},12,8,\overset{4}{\ldots},8)$	D_{12}	(1.14)
$C_2^2\times_{\lambda_2}(C_3\times_\lambda D_8)$	$(96,96,48,32,32,16,12,\overset{4}{\ldots},12,8,\overset{4}{\ldots},8)$	D_{12}	(1.14)
$C_2^2\times_\lambda(C_3\times_\lambda Q_8)$	$(96,96,48,32,32,16,12,\overset{4}{\ldots},12,8,\overset{4}{\ldots},8)$	D_{12}	(1.14)
$C_{13}\times_\lambda C_8$	$(104,104,26,\overset{6}{\ldots},26,8,\overset{6}{\ldots},8)$	C_4	(4.2) [25]
$C_2\times(C_{13}\times_f C_4)$	$(104,104,26,\overset{6}{\ldots},26,8,\overset{6}{\ldots},8)$	C_4	(4.2) [25]
$C_2\times(C_{11}\times_f C_5)$	$(110,110,22,\overset{4}{\ldots},22,10,\overset{8}{\ldots},10)$	C_5	(4.2) [25]
$C_5^2\times_\lambda(C_2\times C_4)$	$(200,50,50,40,40,25,25,10,10,8,\overset{5}{\ldots},8)$	$C_2\times C_4$	(4.8) [25]
$(C_5\times C_2^2)\times_\lambda \mathrm{DC}_3$	$(240,80,60,48,30,16,15,15,8,\overset{4}{\ldots},8,6)$	DC_3	(1.14)
$C_2^4\times_\lambda(C_3^2\times_f C_2)$	$(288,96,96,36,36,32,12,12,9,9,8,8,8,8)$	$C_3^2\times_f C_2$	(1.14)
$C_7^2\times_{f_1}C_6$	$(294,49,\overset{8}{\ldots},49,6,\overset{5}{\ldots},6)$	C_6	(4.2) [25]
$C_7^2\times_{f_3}C_6$	$(294,49,\overset{8}{\ldots},49,6,\overset{5}{\ldots},6)$	C_6	(4.2) [25]
$C_5^2\times_\lambda(C_2\times\Sigma_3)$	$(300,50,\overset{4}{\ldots},50,20,20,10,\overset{4}{\ldots},10,12,6,6)$	D_{12}	(4.2) [25]
$C_7^2\times_f C_8$	$(392,49,\overset{6}{\ldots},49,8,\overset{7}{\ldots},8)$	C_8	(4.8) [25]
$C_{41}\times_f C_{10}$	$(410,41,\overset{4}{\ldots},41,10,\overset{9}{\ldots},10)$	C_{10}	(4.14) [25]
$C_3^3\times_\lambda\Sigma_4$	$(648,108,81,54,24,12,\overset{5}{\ldots},12,9,9,9,6)$	Σ_4	(1.14)
$P\times_f C_5$, $P/C_2^4\simeq C_2^4$	$(1280,256,256,32,\overset{6}{\ldots},32,5,\overset{4}{\ldots},5)$	$C_2^4\times_f C_5$	(1.14)
$C_2^4\times_\lambda \mathrm{SL}(2,5)$	$(1920,1920,128,128,16,\overset{4}{\ldots},16,10,\overset{4}{\ldots},10,6,6)$	A_5	(1.14)
$C_2^5\cdot A_5$	$(1920,320,192,128,16,\overset{4}{\ldots},16,10,\overset{4}{\ldots},10,6,6)$	A_5	(1.14)
$C_{11}^2\times_f(C_5\times_f C_4)$	$(2420,121,\overset{6}{\ldots},121,20,10,10,10,10,4,4)$	$C_5\times_\lambda C_4$	(4.8) [25]
$C_{13}^2\times_f \mathrm{SL}(2,3)$	$(4056,169,\overset{7}{\ldots},169,24,6,\overset{4}{\ldots},6,4)$	$\mathrm{SL}(2,3)$	(4.5) [25]
$C_{17}^2\times_f(\mathrm{SL}(2,3)\cdot C_4)$	$(13872,289,\overset{6}{\ldots},289,48,8,8,8,6,6,4)$	$\mathrm{SL}(2,3)\cdot C_4$	(4.8) [25]

TABLE 4

(i) The finite groups satisfying $r(G)=15$ and $\beta(G)>3$

G	Δ_G	$G/S(G)$	Reference
$C_3^3\times_f C_2$	$(54,27,\overset{13}{\ldots\ldots},27,2)$	C_2	(2.18) [25]
$(C_3\times C_9)\times_f C_2$	$(54,27,\overset{13}{\ldots\ldots},27,2)$	Σ_3	(1.14)
$C_3^4\times_f Q_8$	$(648,81,\overset{10}{\ldots\ldots},81,8,4,4,4)$	Q_8	(4.1) [25]

(ii) The finite groups satisfying $r(G)=15$, $\beta(G)\leqq 3$ and $0\leqq\alpha(G)\leqq 4$

G	Δ_G	$G/S(G)$	Reference
C_{15}	$(15,\overset{15}{\ldots\ldots},15)$	1	(1.16)
$C_3\times D_{14}$	$(42,42,42,21,\overset{9}{\ldots},21,6,6,6)$	C_2	(2.20) [25]
$C_{37}\times_f C_3$	$(111,37,\overset{12}{\ldots\ldots},37,3,3)$	C_3	(2.19) [25]
$(C_3\times C_{13})\times_\lambda C_4$	$(156,78,39,\overset{9}{\ldots},39,12,6,4,4)$	C_4	(4.1) [25]
$(\mathrm{PSL}(2,7)\times C_3)\times_\lambda C_2$	$(1008,504,48,24,24,21,21,21,18,12,12,9,8,8,6)$	C_2	(1.18)
$(A_6\times C_3)\cdot C_4$	$(2160,1180,48,48,27,27,27,24,24,15,15,15,8,8,4)$	C_2	(2.20) [25]

(iii) The finite groups satisfying $r(G)=15$, $\beta(G)\leqq 3$, $5\leqq\alpha(G)\leqq 10$ and $S(G)$ solvable

$C_3\times D_8$	$(24,\overset{6}{\ldots},24,12,\overset{9}{\ldots},12)$	C_2^2	(1.16)
$C_3\times Q_8$	$(24,\overset{6}{\ldots},24,12,\overset{9}{\ldots},12)$	C_2^2	(1.16)
$C_5\times\Sigma_3$	$(30,\overset{5}{\ldots},30,15,\overset{5}{\ldots},15,10,\overset{5}{\ldots},10)$	C_2	(4.2) [25]
$C_3\times\mathrm{Hol}\,C_5$	$(60,60,60,15,15,15,12,\overset{9}{\ldots},12)$	C_4	(4.2) [25]
$C_3\times(C_7\times_f C_3)$	$(63,63,63,21,\overset{6}{\ldots},21,9,\overset{6}{\ldots},9)$	C_3	(4.2) [25]
$C_7\times_\lambda C_9$	$(63,63,63,21,\overset{6}{\ldots},21,9,\overset{6}{\ldots},9)$	C_3	(4.2) [25]
$C_3\times\Sigma_4$	$(72,72,72,24,24,24,12,\overset{6}{\ldots},12,9,9,9)$	Σ_3	(4.2) [25]
$C_3^2\times_{\lambda_1}D_8$	$(72,72,36,\overset{4}{\ldots},36,18,18,12,\overset{6}{\ldots},12,4)$	C_2^2	(4.2) [25]
$C_3^2\times_{\lambda_2}D_8$	$(72,72,36,\overset{4}{\ldots},36,18,18,12,\overset{6}{\ldots},12,4)$	C_2^2	(4.2) [25]
$C_3^2\times_{\lambda_3}Q_8$	$(72,72,36,\overset{4}{\ldots},36,18,18,12,\overset{6}{\ldots},12,4)$	C_2^2	(4.2) [25]
$\Sigma_3\times D_{14}$	$(84,42,\overset{4}{\ldots},42,28,21,21,21,14,14,14,12,6,4)$	C_2^2	(4.2) [25]

TABLE 4 (contd.)

G	Δ_G	$G/S(G)$	Reference
$C_3^3\times_\lambda C_2^2$	$(108,54,54,54,27,\overset{5}{\ldots},27,12,12,12,6,6,6)$	C_2^2	(4.2) [25]
$(C_3\times C_7)\times_\lambda C_6$	$(126,63,42,42,42,21,21,18,18,14,14,9,9,6,6)$	C_6	(4.2) [25]
$C_3^2\times_\lambda D_{16}$	$(144,144,36,\overset{4}{\ldots},36,12,\overset{6}{\ldots},12,8,8,8)$	D_8	(4.2) [25]
$C_3^2\times_\lambda \mathrm{SD}_{16}$	$(144,144,36,\overset{4}{\ldots},36,12,\overset{6}{\ldots},12,8,8,8)$	D_8	(4.2) [25]
$C_3^2\times_\lambda Q_{16}$	$(144,144,36,\overset{4}{\ldots},36,12,\overset{6}{\ldots},12,8,8,8)$	D_8	(4.2) [25]
$C_3^3\times_\lambda D_8$	$(216,108,54,54,27,27,24,12,\overset{6}{\ldots},12,6,6)$	D_8	(4.2) [25]
$C_3^3\times_\lambda C_8$	$(216,108,27,27,27,24,24,24,12,12,12,8,\overset{4}{\ldots},8)$	C_8	(1.14)
$(C_3\times C_{13})\times_\lambda C_6$	$(234,117,39,\overset{6}{\ldots},39,18,18,9,9,6,6,6)$	C_6	(4.2) [25]
$(C_5\times C_3^2)\times_\lambda C_8$	$(360,90,45,\overset{5}{\ldots},45,40,10,8,\overset{6}{\ldots},8)$	C_8	(4.8) [25]
$C_{51}\times_f C_{10}$	$(510,51,\overset{5}{\ldots},51,10,\overset{9}{\ldots},10)$	C_{10}	(4.10) [25]

TABLE 5

(i) The finite groups satisfying $r(G)=16$ and $\beta(G)>4$

G	Δ_G	$G/S(G)$	Reference
C_2^4	$(16,\overset{16}{\ldots\ldots},16)$	1	(1.16)
$C_2^2\times C_4$	$(16,\overset{16}{\ldots\ldots},16)$	C_2	(1.16)
$C_2\times(C_2^4\times_f C_3)$	$(96,96,16,\overset{10}{\ldots\ldots},16,6,\overset{4}{\ldots},6)$	C_3	(4.1) [25]

(ii) The finite groups satisfying $r(G)=16$, $\beta(G)\leqq 4$ and $0\leqq \alpha(G)\leqq 4$

G	Δ_G	$G/S/(G)$	Reference
$C_2^2\times D_{10}$	$(40,\overset{4}{\ldots},40,20,\overset{8}{\ldots},20,8,\overset{4}{\ldots},8)$	C_2	(4.1) [25]
$C_2\times(C_5\times_\lambda C_4)$	$(40,\overset{4}{\ldots},40,20,\overset{8}{\ldots},20,8,\overset{4}{\ldots},8)$	C_2	(4.1) [25]
$C_{13}\times_\lambda C_4$	$(52,52,26,\overset{12}{\ldots\ldots},26,4,4)$	C_2	(2.19) [25]
$C_2\times(C_{13}\times_f C_2)$	$(52,52,26,\overset{12}{\ldots\ldots},26,4,4)$	C_2	(2.19) [25]
$C_{29}\times_f C_2$	$(58,29,\overset{14}{\ldots\ldots},29,2)$	C_2	(2.18) [25]
$C_7^2\times_f C_4$	$(196,49,\overset{12}{\ldots\ldots},49,4,4,4)$	C_4	(2.20) [25]

TABLE 5 (contd.)

(iii) The finite groups satisfying $r(G)=16$, $\beta(G)\leqq 4$, $5\leqq\alpha(G)\leqq 10$ and $S(G)$ solvable

G	Δ_G	$G/S(G)$	Reference
$C_4\times D_{10}$	$(40,\overset{4}{\dots},40,20,\overset{8}{\dots},20,8,\overset{4}{\dots},8)$	C_2^2	(1.14)
$D_{10}\times D_{10}$	$(100,50,\overset{4}{\dots},50,25,\overset{4}{\dots},25,20,20,10,\overset{4}{\dots},10,4)$	C_2^2	(4.2) [25]
$(C_5\times C_2^2)\times_\lambda C_6$	$(120,60,60,40,30,30,24,20,20,15,\overset{4}{\dots},15,8,6,6)$	C_6	(1.14)
$C_2\times(C_{17}\times_f C_4)$	$(136,136,34,\overset{8}{\dots},34,8,\overset{6}{\dots},8)$	C_4	(4.2) [25]
$C_{17}\times_\lambda C_8$	$(136,136,34,\overset{8}{\dots},34,8,\overset{6}{\dots},8)$	C_4	(4.2) [25]
$C_2\times(C_{13}\times_f C_6)$	$(156,156,26,\overset{4}{\dots},26,6,\overset{10}{\dots\dots},6)$	C_6	(1.14)
$C_{13}\times_\lambda C_{12}$	$(156,156,26,\overset{4}{\dots},26,6,\overset{10}{\dots\dots},6)$	C_6	(1.14)
$C_2\times(C_2^4\times_f C_5)$	$(160,160,32,\overset{6}{\dots},32,10,\overset{8}{\dots},10)$	C_5	(4.2) [25]
$(C_2^2\times C_{13})\times_\lambda C_6$	$(312,104,52,\overset{8}{\dots},52,24,8,6,6,6,6)$	C_6	(4.2) [25]
$C_{61}\times_f C_6$	$(366,61,\overset{10}{\dots\dots},61,6,\overset{5}{\dots},6)$	C_6	(4.2) [25]
$C_2\times(C_5^2\times_f Q_8)$	$(400,400,50,\overset{6}{\dots},50,16,16,8,\overset{6}{\dots},8)$	Q_8	(4.2) [25]
$C_5^2\times_\lambda(C_4\times_\lambda C_4)$	$(400,400,50,\overset{6}{\dots},50,16,16,8,\overset{6}{\dots},8)$	Q_8	(4.2) [25]
$C_2^6\times_f C_9$	$(576,64,\overset{7}{\dots},64,9,\overset{8}{\dots},9)$	C_9	(4.2) [25]
$C_2\times(C_5^2\times_f \mathrm{DC}_3)$	$(600,600,50,\overset{4}{\dots},50,24,24,12,\overset{4}{\dots},12,8,\overset{4}{\dots},8)$	DC_3	(1.14)
$C_5^2\times_\lambda(C_3\times_\lambda C_8)$	$(600,600,50,\overset{4}{\dots},50,24,24,12,\overset{4}{\dots},12,8,\overset{4}{\dots},8)$	DC_3	(1.14)
$C_{61}\times_f C_{10}$	$(610,61,\overset{6}{\dots},61,10,\overset{9}{\dots},10)$	C_{10}	(4.14) [25]
$C_2^6\times_{\lambda_1}(C_7\times_f C_3)$	$(1344,192,192,192,64,64,12,\overset{8}{\dots},12,7,7)$	$C_7\times_f C_3$	(1.14)
$C_2^6\times_{\lambda_2}(C_7\times_f C_3)$	$(1344,192,192,192,64,64,12,\overset{8}{\dots},12,7,7)$	$C_7\times_f C_3$	(1.14)
$C_{11}^2\times_f \mathrm{DC}_3$	$(1452,121,\overset{10}{\dots\dots},121,12,6,6,4,4)$	DC_3	(4.2) [25]
$(C_2^2\times C_5^2)\times_\lambda \mathrm{SL}(2,3)$	$(2400,800,100,\overset{4}{\dots},100,96,32,16,\overset{4}{\dots},16,6,\overset{4}{\dots},6)$	$\mathrm{SL}(2,3)$	(1.14)
$C_{13}^2\times_f(C_7\times_\lambda C_4)$	$(4732,169,\overset{6}{\dots},169,28,14,\overset{6}{\dots},14,4,4)$	$C_7\times_\lambda C_4$	(4.14) [25]
$C_{29}^2\times_f \mathrm{SL}(2,5)$	$(100920,841,\overset{7}{\dots},841,120,10,\overset{4}{\dots},10,6,6,4)$	$\mathrm{SL}(2,5)$	(4.11) [25]

TABLE 6

(i) The finite groups satisfying $r(G) = 17$ and $\beta(G) > 5$

$\varnothing$

(ii) The finite groups satisfying $r(G) = 17$, $\beta(G) \leqq 5$ and $0 \leqq \alpha(G) \leqq 4$

G	Δ_G	$G/S(G)$	Reference
C_{17}	$(17,\overset{17}{\ldots\ldots},17)$	1	(1.16)
$C_{31} \times_f C_2$	$(62,31,\overset{15}{\ldots\ldots},31,2)$	C_2	(2.18) [25]
$C_{43} \times_f C_3$	$(129,43,\overset{14}{\ldots\ldots},43,3,3)$	C_3	(2.19) [25]
$C_{61} \times_f C_5$	$(305,61,\overset{12}{\ldots\ldots},61,5,5,5,5)$	C_5	(4.1) [25]
$(A_5 \times C_5) \times_\lambda C_2$	$(600,300,300,40,30,25,\overset{5}{\ldots},25,20,20,15,15,12,6,4)$	C_2	(2.20) [25]

(iii) The finite groups satisfying $r(G) = 17$, $\beta(G) \leqq 5$, $5 \leqq \alpha(G) \leqq 10$ and $S(G)$ solvable

G	Δ_G	$G/S(G)$	Reference
$C_7 \times_{\lambda_1} D_8$	$(56,56,28,\overset{13}{\ldots\ldots},28,4,4)$	C_2^2	(4.2) [25]
$C_7 \times_{\lambda_2} D_8$	$(56,56,28,\overset{13}{\ldots\ldots},28,4,4)$	C_2^2	(4.2) [25]
$C_7 \times_\lambda Q_8$	$(56,56,28,\overset{13}{\ldots\ldots},28,4,4)$	C_2^2	(4.2) [25]
$(C_7 \times C_5) \times_\lambda C_4$	$(140,70,70,70,35,\overset{7}{\ldots},35,28,14,14,14,4,4)$	C_4	(4.2) [25]
$C_2^2 \times_\lambda (C_{21} \times_f C_2)$	$(168,84,84,84,56,28,28,28,21,\overset{7}{\ldots},21,4,4)$	Σ_3	(4.2) [25]
$(C_5 \times C_7) \times_\lambda C_6$	$(210,105,105,35,\overset{5}{\ldots},35,30,30,15,\overset{4}{\ldots},15,6,6,6)$	C_6	(4.2) [25]
$C_7^2 \times_\lambda C_6$	$(294,147,147,49,\overset{7}{\ldots},49,42,14,14,6,6,6,6)$	C_6	(4.2) [25]
$C_{67} \times_f C_6$	$(402,67,\overset{11}{\ldots\ldots},67,6,\overset{5}{\ldots},6)$	C_6	(4.2) [25]
$C_{71} \times_f C_7$	$(497,71,\overset{10}{\ldots\ldots},71,7,\overset{6}{\ldots},7)$	C_7	(4.5) [25]
$C_{73} \times_f C_8$	$(584,73,\overset{9}{\ldots},73,8,\overset{7}{\ldots},8)$	C_8	(4.8) [25]
$C_{73} \times_f C_9$	$(657,73,\overset{8}{\ldots},73,9,\overset{8}{\ldots},9)$	C_9	(4.11) [25]
$C_{71} \times_f C_{10}$	$(710,71,\overset{7}{\ldots},71,10,\overset{9}{\ldots},10)$	C_{10}	(4.14) [25]
$C_{67} \times_f C_{11}$	$(737,67,\overset{6}{\ldots},67,11,\overset{10}{\ldots\ldots},11)$	C_{11}	(1.15)
$C_{31}^2 \times_f \mathrm{SL}(2,5)$	$(115320,961,\overset{8}{\ldots},961,120,10,\overset{4}{\ldots},10,6,6,4)$	SL(2,5)	(4.11) [25]

TABLE 7

(i) The finite groups satisfying $r(G)=18$ and $\beta(G)>6$

G	Δ_G	$G/S(G)$	Reference
$C_3^4\times_f C_8$	$(648,81,\overset{10}{\ldots\ldots},81,8,\overset{7}{\ldots},8)$	C_8	(4.8) [25]

(ii) The finite groups satisfying $r(G)=18$, $\beta(G)\leqq 6$ and $0\leqq\alpha(G)\leqq 4$

G	Δ_G	$G/S(G)$	Reference
$C_2\times C_3^2$	$(18,\overset{18}{\ldots\ldots},18)$	1	(1.16)
$C_3\times(C_3^2\times_f C_2)$	$(54,54,54,27,\overset{12}{\ldots\ldots},27,6,6,6)$	C_2	(2.20) [25]
$(C_3\times C_5)\times_\lambda C_4$	$(60,60,30,\overset{14}{\ldots\ldots},30,4,4)$	C_2	(2.19) [25]
$C_2\times((C_3\times C_5)\times_f C_2)$	$(60,60,30,\overset{14}{\ldots\ldots},30,4,4)$	C_2	(2.19) [25]
$(C_3\times C_{11})\times_f C_2$	$(66,33,\overset{16}{\ldots\ldots},33,2)$	C_2	(2.18) [25]
$C_2\times(C_{19}\times_f C_3)$	$(114,114,38,\overset{12}{\ldots\ldots},38,6,6,6,6)$	C_3	(4.1) [25]
$(C_3\times C_{17})\times_\lambda C_4$	$(204,102,51,\overset{12}{\ldots\ldots},51,12,6,4,4)$	C_4	(4.1) [25]

(iii) The finite groups satisfying $r(G)=18$, $\beta(G)\leqq 6$, $5\leqq\alpha(G)\leqq 10$ and $S(G)$ solvable

G	Δ_G	$G/S(G)$	Reference
$C_6\times\Sigma_3$	$(36,\overset{6}{\ldots},36,18,\overset{6}{\ldots},18,12,\overset{6}{\ldots},12)$	C_2	(4.2) [25]
$C_3\times DC_3$	$(36,\overset{6}{\ldots},36,18,\overset{6}{\ldots},18,12,\overset{6}{\ldots},12)$	C_2	(4.2) [25]
$C_2\times(C_3\times_{\lambda_1}D_8)$	$(48,\overset{4}{\ldots},48,24,\overset{10}{\ldots\ldots},24,8,\overset{4}{\ldots},8)$	C_2^2	(1.14)
$C_2\times(C_3\times_{\lambda_2}D_8)$	$(48,\overset{4}{\ldots},48,24,\overset{10}{\ldots\ldots},24,8,\overset{4}{\ldots},8)$	C_2^2	(1.14)
$C_2\times(C_3\times_\lambda Q_8)$	$(48,\overset{4}{\ldots},48,24,\overset{10}{\ldots\ldots},24,8,\overset{4}{\ldots},8)$	C_2^2	(1.14)
$C_3\times_\lambda((C_4\times C_2)\times_\lambda C_2)$	$(48,\overset{4}{\ldots},48,24,\overset{10}{\ldots\ldots},24,8,\overset{4}{\ldots},8)$	C_2^2	(1.14)
$(C_3\times C_4\times C_2)\times_\lambda C_2$	$(48,\overset{4}{\ldots},48,24,\overset{10}{\ldots\ldots},24,8,\overset{4}{\ldots},8)$	C_2^2	(1.14)
$C_3\times_\lambda(C_4\times_\lambda C_4)$	$(48,\overset{4}{\ldots},48,24,\overset{10}{\ldots\ldots},24,8,\overset{4}{\ldots},8)$	C_2^2	(1.14)
$C_{12}\times_\lambda C_4$	$(48,\overset{4}{\ldots},48,24,\overset{10}{\ldots\ldots},24,8,\overset{4}{\ldots},8)$	C_2^2	(1.14)
$C_3^2\times_\lambda(C_2\times C_4)$	$(72,72,36,\overset{4}{\ldots},36,24,\overset{4}{\ldots},24,18,18,12,\overset{4}{\ldots},12,8,8)$	$C_2\times C_4$	(1.14)
$C_2\times\Sigma_3\times\Sigma_3$	$(72,72,36,\overset{4}{\ldots},36,24,\overset{4}{\ldots},24,18,18,12,\overset{4}{\ldots},12,8,8)$	C_2^2	(1.14)

TABLE 7 (contd.)

G	Δ_G	$G/S(G)$	Reference
$C_3\times(C_3^2\times_f C_4)$	$(108,108,108,27,\overset{6}{\ldots},27,12,\overset{9}{\ldots},12)$	C_4	(4.2) [25]
$C_3^3\times_\lambda C_2^2$	$(108,54,\overset{5}{\ldots},54,36,27,\overset{4}{\ldots},27,18,\overset{4}{\ldots},18,12,6,4)$	C_2^2	(4.2) [25]
$(C_3\times C_5)\times_\lambda C_8$	$(120,120,60,60,30,\overset{6}{\ldots},30,24,24,12,12,8,\overset{4}{\ldots},8)$	C_4	(4.2) [25]
$C_2\times((C_3\times C_5)\times_\lambda C_4)$	$(120,120,60,60,30,\overset{6}{\ldots},30,24,24,12,12,8,\overset{4}{\ldots},8)$	C_4	(4.2) [25]
$C_2\times(C_2^2\times_\lambda D_{18})$	$(144,144,72,72,48,48,24,24,18,\overset{6}{\ldots},18,8,\overset{4}{\ldots},8)$	Σ_3	(1.14)
$C_2\times(C_2^2\times_\lambda(C_3^2\times_f C_2))$	$(144,144,72,72,48,48,24,24,18,\overset{6}{\ldots},18,8,\overset{4}{\ldots},8)$	Σ_3	(1.14)
$(C_5\times C_3^2)\times_\lambda C_4$	$(180,90,90,45,\overset{10}{\ldots\ldots},45,20,10,10,4,4)$	C_4	(4.2) [25]
$C_2\times(C_{19}\times_f C_6)$	$(228,228,38,\overset{6}{\ldots},38,6,\overset{10}{\ldots\ldots},6)$	C_6	(1.14)
$C_{19}\times_\lambda C_{12}$	$(228,228,38,\overset{6}{\ldots},38,6,\overset{10}{\ldots\ldots},6)$	C_6	(1.14)
$C_2^4\times_\lambda(C_9\times_f C_2)$	$(228,144,96,96,96,48,\overset{6}{\ldots},48,9,9,9,8,\overset{4}{\ldots},8)$	Σ_3	(4.2) [25]
$C_2^4\times_\lambda(C_3^2\times_f C_2)$	$(228,144,96,96,96,48,\overset{6}{\ldots},48,9,9,9,8,\overset{4}{\ldots},8)$	Σ_3	(4.2) [25]
$(C_3\times C_{19})\times_\lambda C_6$	$(342,171,57,\overset{9}{\ldots},57,18,18,9,9,6,6,6)$	C_6	(4.2) [25]
$(C_5\times C_3^2)\times_\lambda(C_4\times C_2)$	$(360,90,90,90,72,45,\overset{4}{\ldots},45,40,18,18,10,8,\overset{5}{\ldots},8)$	$C_2\times C_4$	(1.14)
$(C_5\times C_3^2)\times_\lambda Q_8$	$(360,180,180,45,\overset{5}{\ldots},45,40,20,\overset{7}{\ldots},20,4,4)$	Q_8	(1.14)
$(C_3\times C_{17})\times_\lambda C_8$	$(408,204,51,\overset{6}{\ldots},51,24,24,24,12,12,12,8,\overset{4}{\ldots},8)$	C_8	(1.14)
$C_{73}\times_f C_6$	$(438,73,\overset{12}{\ldots\ldots},73,6,\overset{5}{\ldots},6)$	C_6	(4.2) [25]
$(C_3\times C_2^4)\times_\lambda D_{10}$	$(480,240,96,96,96,48,48,48,15,\overset{6}{\ldots},15,8,\overset{4}{\ldots},8)$	D_{10}	(1.14)
$(C_3\times C_5^2)\times_\lambda Q_8$	$(600,300,75,\overset{9}{\ldots},75,24,12,\overset{4}{\ldots},12,4,4)$	Q_8	(4.2) [25]
$C_3^4\times_{f_2} C_8$	$(648,81,\overset{10}{\ldots\ldots},81,8,\overset{7}{\ldots},8)$	C_8	(4.8) [25]
$C_3^4\times_f C_{10}$	$(810,81,\overset{8}{\ldots},81,10,\overset{9}{\ldots},10)$	C_{10}	(4.14) [25
$C_5^2\times_\lambda(C_9\times_\lambda C_4)$	$(900,450,75,\overset{8}{\ldots},75,36,18,\overset{7}{\ldots},18,4,4)$	DC_3	(1.14)
$(C_3\times C_5^2)\times_\lambda DC_3$	$(900,450,75,\overset{8}{\ldots},75,36,18,\overset{7}{\ldots},18,4,4)$	DC_3	(1.14)
$(C_2^2\times C_5^2)\times_\lambda DC_3$	$(1200,400,100,\overset{8}{\ldots},100,48,16,8,\overset{4}{\ldots},8,6,6)$	DC_3	(1.14)
$C_3^4\times_{\lambda_1} SD_{16}$	$(1296,162,\overset{4}{\ldots},162,81,81,81,36,18,\overset{4}{\ldots},18,16,8,8,8,4)$	SD_{16}	(1.14)
$C_3^4\times_{\lambda_2} SD_{16}$	$(1296,162,\overset{4}{\ldots},162,81,81,81,36,18,\overset{4}{\ldots},18,16,8,8,8,4)$	SD_{16}	(1.14)

TABLE 8

(i) The finite groups satisfying $r(G) = 19$ and $\beta(G) > 7$

G	Δ_G	$G/S(G)$	Reference
$C_7^2 \times_{f_1} C_3$	$(147, 49, \overset{16}{\ldots\ldots}, 49, 3, 3)$	C_3	(2.19) [25]

(ii) The finite groups satisfying $r(G) = 19$, $\beta(G) \leqq 7$ and $0 \leqq \alpha(G) \leqq 4$

G	Δ_G	$G/S(G)$	Reference
C_{19}	$(19, \overset{19}{\ldots\ldots}, 19)$	1	(1.16)
$(C_5 \times C_7) \times_f C_2$	$(70, 35, \overset{17}{\ldots\ldots}, 35, 2)$	C_2	(2.18) [25]
$C_7^2 \times_{f_2} C_3$	$(147, 49, \overset{16}{\ldots\ldots}, 49, 3, 3)$	C_3	(2.19) [25]
$C_7^2 \times_{f_3} C_3$	$(147, 49, \overset{16}{\ldots\ldots}, 49, 3, 3)$	C_3	(2.19) [25]
$C_{61} \times_f C_4$	$(244, 61, \overset{15}{\ldots\ldots}, 61, 4, 4, 4)$	C_4	(2.20) [25]
$C_{71} \times_f C_5$	$(355, 71, \overset{14}{\ldots\ldots}, 71, 5, 5, 5, 5)$	C_5	(4.1) [25]

(iii) The finite groups satisfying $r(G) = 19$, $\beta(G) \leqq 7$, $5 \leqq \alpha(G) \leqq 10$ and $S(G)$ solvable

G	Δ_G	$G/S(G)$	Reference
$C_{79} \times_f C_6$	$(474, 79, \overset{13}{\ldots\ldots}, 79, 6, \overset{5}{\ldots}, 6)$	C_6	(4.2) [25]
$C_{89} \times_f C_8$	$(712, 89, \overset{11}{\ldots\ldots}, 89, 8, \overset{7}{\ldots}, 8)$	C_8	(4.8) [25]
$C_{89} \times_f C_{11}$	$(979, 89, \overset{8}{\ldots}, 89, 11, \overset{10}{\ldots\ldots}, 11)$	C_{11}	(1.15)
$C_5^3 \times_\lambda DC_3$	$(1500, 375, 125, \overset{10}{\ldots\ldots}, 125, 30, 15, 15, 12, 6, 4, 4)$	DC_3	(4.2) [25]
$(C_3^2 \times C_5^2) \times_\lambda SL(2,3)$	$(5400, 675, 225, \overset{9}{\ldots}, 225, 24, 18, 18, 9, 9, 6, 6, 4)$	$SL(2,3)$	(4.5) [25]
$C_{17}^2 \times_f SL(2,3)$	$(6936, 289, \overset{12}{\ldots\ldots}, 289, 24, 6, 6, 6, 6, 4)$	$SL(2,3)$	(4.5) [25]
$C_{19}^2 \times_f (C_5 \times_\lambda C_8)$	$(14440, 361, \overset{9}{\ldots}, 361, 40, 10, 10, 8, \overset{6}{\ldots}, 8)$	$C_5 \times_\lambda C_8$	(4.14) [25]
$C_{23}^2 \times_f (SL(2,3) \cdot C_4)$	$(25392, 529, \overset{11}{\ldots\ldots}, 529, 48, 8, 8, 8, 8, 6, 6, 4)$	$SL(2,3) \cdot C_4$	(4.8) [25]

TABLE 9

(i) The finite groups satisfying $r(G)=20$ and $\beta(G)>8$

$\varnothing$

(ii) The finite groups satisfying $r(G)=20$, $\beta(G)\leqq 8$ and $0\leqq\alpha(G)\leqq 4$

G	Δ_G	$G/S(G)$	Reference
$C_2^2\times D_{14}$	$(56,\overset{4}{\ldots},56,28,\overset{12}{\ldots\ldots},28,8,\overset{4}{\ldots},8)$	C_2	(4.1) [25]
$C_2\times(C_7\times_\lambda C_4)$	$(56,\overset{4}{\ldots},56,28,\overset{12}{\ldots\ldots},28,8,\overset{4}{\ldots},8)$	C_2	(4.1) [25]
$C_{17}\times_\lambda C_4$	$(68,68,34,\overset{16}{\ldots\ldots},34,4,4)$	C_2	(2.19) [25]
$C_2\times(C_{17}\times_f C_2)$	$(68,68,34,\overset{16}{\ldots\ldots},34,4,4)$	C_2	(2.19) [25]
$C_{37}\times_f C_2$	$(74,37,\overset{18}{\ldots\ldots},37,2)$	C_2	(2.18) [25]
$(C_2^2\times C_{13})\times_f C_3$	$(156,52,\overset{17}{\ldots\ldots},52,3,3)$	C_3	(2.19) [25]
$(C_5\times C_{13})\times_f C_4$	$(260,65,\overset{16}{\ldots\ldots},65,4,4,4)$	C_4	(2.20) [25]
$C_{11}^2\times_f Q_8$	$(968,121,\overset{15}{\ldots\ldots},121,8,4,4,4)$	Q_8	(4.1) [25]

(iii) The finite groups satisfying $r(G)=20$, $\beta(G)\leqq 8$, $5\leqq\alpha(G)\leqq 10$ and $S(G)$ solvable

G	Δ_G	$G/S(G)$	Reference
$C_2^2\times C_5$	$(20,\overset{20}{\ldots\ldots},20)$	1	(1.16)
$C_5\times D_{10}$	$(50,\overset{5}{\ldots},50,25,\overset{10}{\ldots\ldots},25,10,\overset{5}{\ldots},10)$	C_2	(4.2) [25]
$C_5\times A_4$	$(60,\overset{5}{\ldots},60,20,\overset{5}{\ldots},20,15,\overset{10}{\ldots\ldots},15)$	C_3	(4.2) [25]
$D_{10}\times D_{14}$	$(140,70,\overset{5}{\ldots},70,35,\overset{6}{\ldots},35,28,20,14,14,14,10,10,4)$	C_2^2	(4.2) [25]
$C_2^4\times_\lambda D_{12}$	$(192,192,64,\overset{6}{\ldots},64,32,32,16,\overset{8}{\ldots},16,6,6)$	Σ_3	(1.14)
$C_2^4\times_\lambda \mathrm{D}C_3$	$(192,192,64,\overset{6}{\ldots},64,32,32,16,\overset{8}{\ldots},16,6,6)$	Σ_3	(1.14)
$C_2\times(C_5^2\times_f C_4)$	$(200,200,50,\overset{12}{\ldots\ldots},50,8,\overset{6}{\ldots},8)$	C_4	(4.2) [25]
$C_5^2\times_\lambda C_8$	$(200,200,50,\overset{12}{\ldots\ldots},50,8,\overset{6}{\ldots},8)$	C_4	(4.2) [25]
$C_7^2\times_\lambda \Sigma_3$	$(294,98,\overset{6}{\ldots},98,49,\overset{5}{\ldots},49,14,\overset{7}{\ldots},14,3)$	Σ_3	(4.2) [25]
$C_2\times(C_5^2\times_f C_6)$	$(300,300,50,\overset{8}{\ldots},50,12,\overset{10}{\ldots\ldots},12)$	C_6	(1.14)
$C_5^2\times_\lambda C_{12}$	$(300,300,50,\overset{8}{\ldots},50,12,\overset{10}{\ldots\ldots},12)$	C_6	(1.14)

TABLE 9 (contd.)

G	Δ_G	$G/S(G)$	Reference
$C_7^2 \times_\lambda D_8$	$(392, 98, \overset{6}{\ldots}, 98, 49, 49, 49, 28, 28, 14, \overset{6}{\ldots}, 14, 8, 4)$	D_8	(1.14)
$(C_2^2 \times C_{19}) \times_\lambda C_6$	$(456, 152, 76, \overset{12}{\ldots\ldots}, 76, 24, 8, 6, 6, 6, 6)$	C_6	(4.2) [25]
$(C_5 \times C_{17}) \times_\lambda C_8$	$(680, 170, 85, \overset{10}{\ldots\ldots}, 85, 40, 10, 8, \overset{6}{\ldots}, 8)$	C_8	(4.8) [25]
$C_{97} \times_f C_8$	$(776, 97, \overset{12}{\ldots\ldots}, 97, 8, \overset{7}{\ldots}, 8)$	C_8	(4.8) [25]
$C_{101} \times_f C_{10}$	$(1010, 101, \overset{10}{\ldots\ldots}, 101, 10, \overset{9}{\ldots}, 10)$	C_{10}	(4.14) [25]
$C_2 \times (C_7^2 \times_f DC_3)$	$(1176, 1176, 98, \overset{8}{\ldots}, 98, 24, 24, 12, \overset{4}{\ldots}, 12, 8, \overset{4}{\ldots}, 8)$	DC_3	(1.14)
$C_7^2 \times_\lambda (C_3 \times_\lambda C_8)$	$(1176, 1176, 98, \overset{8}{\ldots}, 98, 24, 24, 12, \overset{4}{\ldots}, 12, 8, \overset{4}{\ldots}, 8)$	DC_3	(1.14)
$C_{13}^2 \times_f DC_3$	$(2028, 169, \overset{14}{\ldots\ldots}, 169, 12, 6, 6, 4, 4)$	DC_3	(4.2) [25]
$(C_2^2 \times C_7^2) \times_\lambda SL(2,3)$	$(4704, 1568, 196, \overset{8}{\ldots}, 196, 96, 32, 16, \overset{4}{\ldots}, 16, 6, \overset{4}{\ldots}, 6)$	$SL(2,3)$	(1.14)

REMARK. In [25] Table 3, The following group is missing:

G	Δ_G	$G/S(G)$
$\mathrm{Hol}(2^5\Gamma_5 a_2, C_3)$	$(96, 96, 16, \overset{5}{\ldots}, 16, 6, \overset{4}{\ldots}, 6)$	$C_2^4 \times_f C_3$

2. Preliminaries

We will often use the preliminary lemmas of [25]. Also we utilize the following lemmas:

LEMMA 1.1. *Let N be a normal subgroup of G such that $G = N \times_\lambda T$. Then:*
(1) $r_G(T) = r(T)$,
(2) $r_G(nT) \geqq r(T)$ *for each* $n \in N$.

PROOF. (1) Set $T = \dot{\bigcup}_{i=1}^{t} \mathrm{Cl}_T(h_i)$. We have $\bigcup_{g \in G} T^g = \bigcup_{i=1}^{t} \mathrm{Cl}_G(h_i)$, and if h_i is conjugate to h_j in G, then there exists $nh \in NT$ such that $h_i^{nh} = h_j$, with $n \in N$ and $h \in T$, therefore $h_i^{-1}h_j^{h^{-1}} = h_i^{-1}h_i^{n} = [h_i, n] \in N \cap T = 1$, i.e. $\mathrm{Cl}_T(h_i) = \mathrm{Cl}_T(h_j)$ and $i = j$. Thus $r_G(T) = r(T)$.

(2) This result is an immediate consequence of the fact that $nh \sim_G n'h'$, $n, n' \in N$, $h, h' \in T$, implies $h \sim_T h'$.

LEMMA 1.2. *If T is a nilpotent S_π-subgroup of g, then G has a normal π-complement if and only if $r_G(T)=r(T)$. In particular, if $\pi=\{p\}$ and P is a Sylow p-subgroup of G, then G has a normal p-complement iff $r_G(P)=r(P)$.*

PROOF. The non-trivial implication follows from [8] corollary 12.5 (p. 102).

LEMMA 1.3. *Let P be a Sylow p-subgroup of G. Then we have the following affirmations:*

(1) $r_G(C_G(P))=r_{N_G(P)}(C_G(P))$.

(2) $|\mathrm{Cl}_G(c)|=\nu_p(G)\cdot|\mathrm{Cl}_{N_G(P)}(x)|\cdot(1/|C_G(x): C_{N_G(P)}(x)|)$ *for each* $x\in N_G(P)$.

(3) *If P is abelian, $N_G(P)=P\times_\lambda T$ and $C_G(P)=P\times T_1$ with $T_1\leqq T$, then we have $T_1\trianglelefteq N_G(P)$ and*

$$r_{N_G(P)}(C_G(P))=r_{N_G(P)}(T_1^*)+r_{N_G(P)}(P)+r_{N_G(P)}(P^*)\cdot r_{N_G(P)}(T_1^*).$$

Furthermore, if $P\leqq Z(N_G(P))$, then $r_G(N_G(P))=r(N_G(P))=|P|\cdot r(T)$.

PROOF. These results are immediate consequences of a well-known theorem of Burnside (cf. [7] Theorem 1.1, p. 240).

REMARK. When P is an abelian group, the analysis of $\Delta_{N_G(P)}$ is developed using Lemma 2.11 of [25].

LEMMA 1.4. *Let G be a group whose elements have primary power orders. Let $|G|=p_1^{a_1}\cdots p_t^{a_t}$ be the decomposition in primes factors of the order of G, with $p_i\neq p_j$ for each $i\neq j$, and let P_i be a Sylow p_i-subgroup of G for every $i=1,\ldots,t$. Then G has exactly $(|Z(P_i)|-1)/(|N_G(P_i)/P_i|)$ conjugacy classes of cardinality $|G/P_i|$ for each $i=1,\ldots,t$. In particular, if the Sylow subgroups P_i are abelian, then*

$$r(G)=1+\sum_{i=1}^{t}(|Z(P_i)|-1)/(|N_G(P_i)/P_i|).$$

PROOF. Let $P\in\mathrm{Syl}_p(G)$. The condition that G does not have elements non-divisible by two primes numbers order implies that $C_G(P)$ is a p-subgroup of G and that if $N_G(P)=P\times_\lambda T$, then T acts f.p.f. over P, that is, $N_G(P)=P\times_f T$. Since $C_G(P)\trianglelefteq N_G(P)$ and $N_G(P)$ is a Frobenius group of kernel P, it follows that either $C_G(P)\leqq P$ or $P<C_G(P)$, consequently $C_G(P)=Z(P)$. Moreover, for each $x\in Z(P)^*$, we have $|\mathrm{Cl}_{N_G(P)}(x)|=|x^T|=|T|$, so

$$r_G(Z(P))=r_G(C_G(P))=r_{N_G(P)}(C_G(P))=1+(|Z(P)|-1)/|N_G(P)/P|,$$

but $\mathrm{Cl}_G(y)\cap Z(P)^*\neq\varnothing$ iff $P^g\leqq C_G(y)$ for some $g\in G$, that is, if $|C_G(y)|=$

$p^a = |P|$. Therefore $r_G(Z(P)) - 1 = (|Z(P)| - 1)/(|N_G(P)/P|)$ is the number of conjugacy classes of elements of G whose cardinality is $|G|/p^a$.

EXAMPLES. (1) By observing the orders of elements of A_5, it is immediate that $N_{A_5}(C_5) \simeq D_{10}$, $N_{A_5}(C_3) \simeq \Sigma_3$ and $N_{A_5}(C_2^2) \simeq A_4$. Then, if $|P_1| = 5$, $|P_2| = 3$ and $|P_3| = 4$, we have

$$r(A_5) = 1 + (5-1)/(5\cdot 2/5) + (3-1)/(3\cdot 2/3) + (4-1)/(4\cdot 3/4)$$
$$= 5.$$

(2) Set $G = \mathrm{PSL}(2,7)$. Then we have $N_G(C_7) = C_7 \times_f C_3$, $N_G(C_3) \simeq \Sigma_3$ and $N_G(D_8) \simeq D_8$, so G has

$(7-1)/(7\cdot 3/7) = 2$ conjugacy classes of cardinality $168/7 = 24$,

$(2-1)/(8/8) = 1$ conjugacy classes of cardinality $168/8 = 21$,

$(3-1)/(6/3) = 1$ conjugacy classes of cardinality $168/3 = 56$.

(3) Consider the group $G = C_2^4 \times_\lambda A_5$ with A_5 acting transitively over C_2^4. Then $N_G(C_5) \simeq D_{10}$, $N_G(C_3) \simeq \Sigma_3$ and if P is a Sylow 2-subgroup of G, then we have $N_G(P) = P \times_\lambda C_3 = C_2^4 \times_\lambda A_4$. Thus G has

$(5-1)/(5\cdot 2/5) = 2$ conjugacy classes of cardinality $|G|/5$,

$(3-1)/(6/3) = 1$ conjugacy classes of cardinality $|G|/3$,

$(4-1)/(2^6\cdot 3/2^6) = 1$ conjugacy classes of cardinality $|G|/2^6$.

Assume the hypothesis of Lemma 1.4; in general, non-abelian Sylow subgroups can exist. Now if $x \in G^*$ and $o(x) = p^e$, with p prime, then $C_G(x)$ is a p-group, so there exists $P \in \mathrm{Syl}_p(G)$ such that $C_G(x) \leqq P$. Consequently $|C_G(x)| = |C_P(x)|$ and $|\mathrm{Cl}_G(x)| = |\mathrm{Cl}_P(x)| \cdot |G/P|$, that is, the cardinal of a conjugacy class of G which is different from $|G/P|$ depends only on Δ_P. Thus, the possible values of the tuple Δ_G are bounded if we know previously Δ_P, when P is any Sylow subgroup of G. In general, we will write

$$r(G) = 1 + \sum_{i=1}^{t} r_G(Z(P_i)^*) + \sum_{i=1}^{t} r_G^*(P_i - Z(P_i))$$

in which we define $r_G^*(P_i - Z(P_i)) = r_G(P_i - Z(P_i)) - \mu_{P_i}$ with

$$\mu_{P_i} = |\{\mathrm{Cl}_G(g) \mid \mathrm{Cl}_G(g) \cap Z(P_i) \neq \varnothing = \mathrm{Cl}_G(g) \cap (P_i - Z(P_i))\}|,$$

that is,

$$r(G) = 1 + \sum_{i=1}^{t} (|Z(P_i)| - 1)/(|N_G(P_i)/P_i|) + \sum_{i=1}^{t} r_G^*(P_i - Z(P_i)).$$

Naturally $r_G(P_i - Z(P_i)) \leqq r_{P_i}(P_i - Z(P_i))$.

EXAMPLES. (1) Consider the group $G = \mathrm{PSL}(2,7)$. Let $P \simeq D_8$ a Sylow 2-subgroup of G. Then $\Delta_P = (8,8,4,4,4)$, so $\Delta^{D_8}_{D_8 - Z(D_8)} = (4,4,4)$ and we have

$$168 = 1 + 168/8 + 168/7 + 168/3 + \sum_{i=1}^{s} 7 \cdot 3 \cdot 8/2^{m_i} \qquad \text{with } 2^{m_i} = 4$$

for each i, consequently $s = 1$ and $\Delta_{\mathrm{PSL}(2,7)} = (168,8,7,7,4,3)$.

(2) Consider the group $G = M_9 = \mathrm{PGL}^*(2,9)$, which is the unique extension of $\mathrm{PSL}(2,9)$ by C_2 with a 2-Sylow of the type SD_{16}. We have

$$N_{M_9}(C_5) \simeq C_5 \times_f C_4, \quad N_{M_9}(C_3^2) \simeq C_3^2 \times_f Q_8 \quad \text{and} \quad N_{M_9}(\mathrm{SD}_{16}) \simeq \mathrm{SD}_{16},$$

therefore M_9 has a unique conjugacy class of elements of order 5, a unique conjugacy class of elements of order 3 and a unique conjugacy class of elements of order 2 that are central in a 2-Sylow of M_9. We have

$$\Delta^{\mathrm{SD}_{16}}_{\mathrm{SD}_{16} - Z(\mathrm{SD}_{16})} = (8,8,8,4),$$

so we consider the equations:

$$(1) \qquad \begin{aligned} 720 &= 1 + 720/16 + 720/9 + 720/5 + 9 \cdot 5 \cdot 2 \cdot t_1 + 9 \cdot 5 \cdot 4 \cdot t_2, \\ r(G) &= 4 + t_1 + t_2. \end{aligned}$$

(1) implies $5 = t_1 + 2t_2$, hence $(t_1, t_2) \in \{(1,2),(3,1)\}$ and the cardinals of the centralizers of the elements of these possible classes are $(8,4,4)$ and $(8,8,8,4)$, being $r(G) = 7$ or 8, respectively. On the other hand, $\Delta_{A_6} = (360,9,9,8,5,5,4)$, hence $r(M_9) = 2 \cdot s + (7 - s)/2$ with $s \geqq 3$ (cf. [25] Lemma 2.9), therefore $r(M_9) \geqq 8$ and necessarily $r(M_9) = 8$. Thus

$$\Delta_{M_9} = ((720,16,9,5),(8,8,8,4)) = (720,16,9,8,8,8,5,4).$$

(3) Let us consider the group $G = C_2^4 \times_\lambda A_5$ with A_5 acting transitively over C_2^4, let $P \in \mathrm{Syl}_2(G)$, then $\Delta^P_{P - Z(P)} = (16, \overset{15}{\ldots}, 16)$. Now observing the equations

$$16 \cdot 60 = 1 + 960/5 + 960/5 + 960/3 + 960/2^6 + (960/16) \cdot t$$

and

$$r(G) = 5 + t,$$

it follows that $t = 4$ and $\Delta_G = (960,64,16,16,16,16,5,5,3)$.

LEMMA 1.5. *Set* $G = \mathrm{P\Gamma L}(2,9)$. *Then we have*

$$\Delta_G = (1440,48,40,32,18,16,16,10,10,8,8,8,6).$$

$r(G) = 13$, $\beta(G) = 1$, $G/S(G) \simeq C_2^2$, $S(G) \simeq A_6$ *and* $\alpha(G) = 8$.

PROOF. We know that $G/A_6 \simeq C_2^2$ and that G has exactly three normal subgroups of index 2: $N_1 \simeq \Sigma_6$, $N_2 \simeq \mathrm{PGL}(2,9)$ and $N_3 \simeq M_9$. Besides

$$\Delta_{N_1} = (720, 48, 48, 18, 18, 16, 8, 8, 6, 6, 5),$$

$$\Delta_{N_2} = (720, 20, 16, 10, 10, 10, 10, 9, 8, 8, 8),$$

$$\Delta_{N_3} = (720, 16, 9, 8, 8, 8, 5, 4).$$

Obviously, $r(G) = r_G(S(G)) + r_G(N_1 - S(G)) + r_G(N_2 - S(G)) + r_G(N_3 - S(G))$ and we have $r(G) = 2s_i + (r(N_i) - s_i)/2$, where s_i is the number of conjugacy classes of N_i fixed by the automorphism ψ_i: $N_i \to N_i$ defined by $\psi_i(x) = x^{g_i}$ for each $x \in N_i$, with g_i an element of G such that $g = N_i\langle g_i\rangle$.

We have $N_1 = S(N_1) \cup (N_1 - S(N_1))$, $\Delta^{N_1}_{S(N_1)} = (720, 18, 18, 16, 8, 5)$ and $\Delta^{N_1}_{N_1 - S(N_1)} = (48, 48, 8, 6, 6)$, so $s_1 \geqq 5$ and $r(G) \in \{13, 16, 19, 22\}$. In [16] it is proved that $s_1 = 5$ and now it is immediate to conclude that $\Delta^G_{S(G)} = (1440, 32, 19, 16, 10)$, $\Delta^G_{N_1 - S(G)} = (48, 16, 6)$, $\Delta^G_{N_2 - S(G)} = (8, 8)$ and $\Delta^G_{N_3 - S(G)} = (40, 10, 8)$. Thus we obtain

$$\Delta_G = (1440, 48, 40, 32, 18, 16, 16, 10, 10, 8, 8, 8, 6) \quad \text{and} \quad \alpha(G) = 13 - 5 = 8.$$

LEMMA 1.6. (1) *If G is a group such that* $\mathrm{PSL}(3,4) \lhd G \leqq \mathrm{Aut}(\mathrm{PSL}(3,4))$, *then* $r(G) \geqq 14$.

(2) $\Delta_{\mathrm{PGL}(2,11)} = (20160, 24, 20, 12, \overset{5}{\ldots}, 12, 11, 10, \overset{4}{\ldots}, 10)$, $r(\mathrm{PGL}(2,11)) = 13$ *and* $\alpha(G) = 6$.

PROOF. These results rely on simple matrix calculations and using the tuples $\Delta_{\mathrm{PSL}(3,4)} = (20160, 64, 16, 16, 16, 9, 7, 7, 5, 5)$, $\Delta_{\mathrm{PSL}(2,11)} = (660, 12, 11, 11, 6, 6, 5, 5)$ and Lemma 2.9(iii) and (iv) from [25].

Let Γ be the family of all finite nilpotent groups. We define $\psi_{11} = \Phi_{11} \cap \Gamma$.

LEMMA 1.7.

$$\psi_{11} = 2^5\Gamma_4 \cup \{2^5\Gamma_3 a_i \mid 1 \leqq i \leqq 3\} \cup \{2^5\Gamma_3 c_i \mid 1 \leqq i \leqq 2\} \cup \{2^5\Gamma_3 d_i \mid 1 \leqq i \leqq 2\}.$$

PROOF. Cf. [24].

In Lemmas 2.18, 2.19 and 2.20 from [25], all finite groups satisfying $1 \leqq \alpha(G) \leqq 3$ are classified. In Lemma 4.1 from [25], we obtain the finite groups satisfying $\alpha(G) = 4$ and with $S(G)$ solvable. In the following, we will obtain all finite groups satisfying $\alpha(G) = 4$.

LEMMA 1.8. *Let G be a finite group with* $S(G)$ *non-solvable and satisfying* $\alpha(G) = 4$. *Then either* $G = \mathrm{PGL}(2,7)$ *or* $G = (\mathrm{PSL}(2,7) \times H) \times_\lambda C_2$ *with* $\mathrm{PSL}(2,7)C_2 = \mathrm{PGL}(2,7)$ *and* $H \times_\lambda C_2 = H \times_f C_2$, *being* $r(G) = 6 + 3|H|$.

PROOF. We have $r(G/S(G)) \leqq 5$. If $r(G/S(G)) = 5 = \alpha(G)+1$, then $|C_G(x)| = |C_{\bar{G}}(\bar{x})|$ for each $x \in G - S(G)$ and $S(G)$ is solvable by Lemma 2.3 from [25], that is impossible.

If $r(G/S(G)) = 4$, then $G/S(G)$ is isomorphic to one of the groups C_4, C_2^2, D_{10}, and A_4.

Suppose $G/S(G) = \bar{G} = \langle \bar{a} \rangle \simeq C_4$. Then

$$\alpha(G) = 4 = r_G(aS(G)) + r_G(a^{-1}S(G)) + r_G(a^2S(G))$$

forces that $r_G(aS(G)) = 1$, hence $C_G(a) = \langle a \rangle$ is isomorphic to C_4 and $S(G)$ is solvable, impossible.

Suppose $\bar{G} = \langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle \simeq C_2^2$. Then

$$4 = r_G(a_1S(G)) + r_G(a_2S(G)) + r_G(a_1a_2S(G))$$

implies that $r_G(aS(G)) = 1$ for some $a \in \{a_1, a_2, a_1a_2\}$, hence $|C_G(a)| = 4$ and if P is a 2-Sylow subgroup of G, then there is $\langle b \rangle \trianglelefteq P$ such that $P/\langle b \rangle \simeq C_2$. We have $o(\bar{b}) = 2$. hence $b^2 \in S(G)$ and $S(G)$ has cyclic Sylow's 2-subgroups, so $S(G)$ is solvable, impossible.

Assume $\bar{G} = \langle \bar{a} \rangle \times_f \langle \bar{b} \rangle \simeq D_{10}$. Then

$$4 = r_G(aS(G)) + r_G(a^2S(G)) + r_G(bS(G))$$

and we have $r_G(aS(G)) = 1$, so $C_G(a) = \langle a \rangle \simeq C_5$ acts f.p.f. over $S(G)$, therefore $S(G)$ is solvable, impossible.

If $\bar{G} = \langle \bar{a}_1, \bar{a}_2 \rangle \times_f \langle \bar{b} \rangle \simeq A_4$, then $r_G(bS(G)) = 1$, hence $C_G(b) \simeq C_3$ and $S(G)$ is solvable impossible. Thus $r(\bar{G}) \leqq 3$ and $\bar{G}$ is isomorphic to one of the following groups: Σ_3, C_3, or C_2.

If $\bar{G} = \langle \bar{a} \rangle \times_f \langle \bar{b} \rangle \simeq \Sigma_3$, then $4 = r_G(aS(G)) + r_G(bS(G))$ and $S(G)$ non-solvable implies $r_G(aS(G)) = 2 = r_G(bS(G))$, hence $|C_G(b)| = 4$ and again $S(G)$ has cyclic 2-Sylow, that is impossible.

If $\bar{G} = \langle \bar{b} \rangle \simeq C_3$, then $r_G(bS(G)) = 2$ and $\Delta_b = (6,6)$, hence Lemma 2.13(ii) from [25] implies that $S(G)$ is solvable.

Thus we conclude that $G/S(G)$ is isomorphic to C_2. If there exists $g \in G - S(G)$ such that $o(g) = 2^e$ and $|C_G(g)| = 2^n \cdot m$ with $n \leqq 3$, then G has sectional range at most 4 and necessarily either $G = \mathrm{PSL}(2,7)$ or $G = (\mathrm{PSL}(2,7) \times H) \times_\lambda C_2$ (cf. [18]). Assume that G has sectional range greater than or equal to 5, and let g be a 2-element in $G - S(G)$. Now, we consider the equation:

$$1/2 = 1/2\lambda_1 + 1/2\lambda_2 + 1/2\lambda_3 + 1/2\lambda_4 \qquad \text{with } \Delta_g = (2\lambda_1, \ldots, 2\lambda_4).$$

If $2\lambda_4 \geqq 8$, then $\Delta^G_{G-S(G)} = (8,8,8,8)$, impossible, hence $2\lambda_4 = 6$. If $2\lambda_3 \geqq 12$, then $1/2 \leqq 1/6 + 3/12$, impossible too, hence $2\lambda_3 = 6$ and $\Delta_g = (24,8,6,6)$ or $(12,12,6,6)$, therefore $|C_G(g)| = 2^n \cdot m$ with $n \leqq 3$, which is impossible.

REMARKS. (1) If A is a non-abelian simple normal subgroup of G and suppose that $G = (A \times H) \times_\lambda C_2 = (A \times H) \times_\lambda \langle b \rangle$ with $HC_2 = H \times_f C_2$ and $AC_2 \neq A \times C_2$, then $\alpha(G) = \alpha(AC_2)$ and $r(G) = 2s + (r(A)|H| - s)/2$, where s is the number of conjugate classes $\mathrm{Cl}_A(a)$ of A such that $\mathrm{Cl}_A(a)^b = \mathrm{Cl}_A(a)$, i.e. $s = \alpha(AC_2)$ (it is an immediate consequence of [25] Lemma 2.9).

(2) If $G/S(G) = \langle \bar{g} \rangle \simeq C_p$, with p prime, then we have $\alpha(G) = s \cdot (p-1)$, where s is the number of conjugacy classes of G fixed by the automorphism $\psi: S(G) \to S(G)$ defined by $\psi(x) = x^g$ for each $x \in S(G)$. In particular, $\alpha(G) = s$, in case $p = 2$.

LEMMA 1.9. *Let V be a vector space over Z_p of dimension n and let $f \in \mathrm{Aut}_{Z_p}(V)$ be such that $f^{p^t} = 1$ for some $t \in \mathbf{N}$. Then $|C_V(f)| \geqq p^e$, with e a natural number satisfying $e \geqq n/m \geqq n/p^t$, where m is the degree of the minimal polynomial of f over Z_p. In particular, if $p = 2$ and $o(f) = 2$, then $|C_V(f)| \geqq 2^k$ if $n = 2k$, and $|C_V(f)| \geqq 2^{k+1}$ if $n = 2k+1$ for some natural number k.*

PROOF. We know that there exist f-invariable subspaces $V_1, \ldots, V_s$ of V and polynomials $q_1(x), \ldots, q_s(x) \in Z_p[x]$ such that $V = V_1 \oplus \cdots \oplus V_s$, $q_i(x)$ divides $q_{i+1}(x)$ for each $i = 1, \ldots, s-1$, $q_s(x)$ is the minimal polynomial of f, $q_i(x) =$ pol. min.$(f_{|V_i})$ and $q_1(x)q_2(x)\cdots q_s(x)$ is the characteristic polynomial of f. As f is a root of the polynomial $x^{p^t} - 1 = (x-1)^{p^t}$, the minimal polynomial pol. min.(f) divides $(x-1)^{p^t}$, so $m \leqq p^t$.

Let us consider the p-group $G = \mathrm{Hol}(V, \langle f \rangle)$. We have $V_i \trianglelefteq G$ for each i, hence $V_i \cap Z(G) \neq 1$ and therefore $|C_{V_i}(f)| \geqq p$ for every i. In consequence $|C_V(f)| \geqq p^s$. Besides

$$1 \leqq \mathrm{degr.}(q_1(x)) \leqq \cdots \leqq \mathrm{degr.}(q_s(x)) = m \leqq p^t$$

and

$$\mathrm{degr.}(q_1) + \cdots + \mathrm{degr.}(q_s) = n,$$

hence $n \leqq s \cdot \mathrm{degr.}(q_s) = sm$, i.e. $s \geqq n/m$.

EXAMPLE. Suppose $f \in \mathrm{Aut}(C_3^4)$ and $o(f) = 3$, then $|C_{C_3^4}(f)| \geqq 3^e$, with $e \geqq 4/3$, so $e \geqq 2$ and $|C_{C_3^4}(f)| \geqq 3^2$.

LEMMA 1.10. *Let G be a group with $S(G)$ abelian and let $x \in G - S(G)$. Put $\bar{G} = G/S(G)$. Then $r_G(xS(G)) \geqq o(\bar{x}) \cdot |C_G(x) \cap S(G)| / |C_{\bar{G}}(\bar{x})|$.*

PROOF. Let $\mathrm{Cl}_G(xz_j)$, $j = 1, \ldots, t$ be the conjugacy classes of elements of G which have non-empty intersection with $xS(G)$. Then $t = r_G(xS(G))$ and $1/|C_{\bar{G}}(\bar{x})| = \Sigma_{j=1}^{t} 1/|C_G(xz_j)|$ (cf. [25] Lemma 2.1(ii)). Moreover, $o(\bar{x}\bar{z}_j) = o(\bar{x})$ and $C_G(xz_j) \cap S(G) = C_G(x) \cap S(G)$, because $S(G)$ is an abelian group, therefore

$$|C_G(xz_j)| \geqq o(\bar{x}) \cdot |C_G(x) \cap S(G)| \qquad \text{for every } j$$

and consequently $t \geqq o(\bar{x}) \cdot |C_G(x) \cap S(G)|/|C_{\bar{G}}(\bar{x})|$.

Lemma 1.10 is generally used with Lemma 1.9, fixing the possible values of $r_G(xS(G))$, then the cardinal of $C_G(x) \cap S(G)$ is bounded, and if $o(\bar{x})$ is the power of a prime number p, the situations that originate from fixing the possible orders of $C_G(x) \cap O_p(S(G))$ $(= C_G(x) \cap S(G))$ are now analyzed.

LEMMA 1.11. *Let G be a finite group and let $S_1, \ldots, S_n$ be normal sets of G. Then*

$$r_G\left(\bigcup_{i=1}^{n} S_i\right) = \sum_{t=1}^{n} \sum_{1 \leqq i_1 < \cdots < i_t \leqq n} r_G\left(\bigcap_{k=1}^{t} S_{i_k}\right)(-1)^{t+1}.$$

PROOF. This result follows immediately from an inductive process over n and from the fact that $r_G(S_1 \cup S_2) = r_G(S_1) + r_G(S_2) - r_G(S_1 \cap S_2)$.

LEMMA 1.12. *Let G be a group such that $S(G)$ is abelian. Set*

$$\bar{G} = G/S(G) = \mathrm{Cl}_{\bar{G}}(\bar{x}_1) \dot{\cup} \cdots \dot{\cup} \mathrm{Cl}_{\bar{G}}(\bar{x}_n) \quad \textit{and} \quad \mathrm{Cl}_{\bar{G}}(\bar{x}_i) = \{\bar{x}_{i_1}, \ldots, \bar{x}_{i_{n_i}}\}.$$

Then $S_i = (C_G(x_{i_1}) \cap S(G)) \cup \cdots \cup ((C_G(x_{i_{n_i}}) \cap S(G))$ is a normal set in G and

$$r(G) = \alpha(G) + r_G\left(\bigcup_{i=1}^{n} S_i\right) + \left(|S(G)| - \left|\bigcup_{i=1}^{n} S_i\right|\right) \Big/ |G/S(G)|.$$

PROOF. Let g be an element of G and set $\bar{x}_{i_j}^{\bar{g}} = \bar{x}_{i_k}$, then $x_{i_j}^{g} = x_{i_k} \cdot z$ for some $z \in S(G)$ and $(C_G(x_{i_j}) \cap S(G))^g = C_G(x_{i_k}z) \cap S(G) = C_G(x_{i_k}) \cap S(G)$. Therefore S_i is a normal set in G. Besides, if $z \in S(G) - \bigcup_{i=1}^{n} S_i$, then $z^a = z^b$ with $a, b \in G - S(G)$ if and only if $z \in C_G(ab^{-1}) \cap S(G)$, so $\bar{a}\bar{b}^{-1} = \bar{1}$ and $aS(G) = bS(G)$. Therefore $|\mathrm{Cl}_G(z)| = |G/S(G)|$ and thus we get the desired formula.

Lemmas 1.11 and 1.12 are generally used to determine $r(G)$, once the value of $\alpha(G)$ has been fixed.

LEMMA 1.13. *Let G be a finite group such that $S(G)$ is not solvable and $\beta(G) = r(G) - j$ with $1 \leqq j \leqq 11$. Then G is isomorphic to one of the following groups*: A_5, A_6, A_7, Σ_5, Σ_6, $A_5 \times C_2$, $\mathrm{PSL}(2,7) \times C_2$, $\mathrm{PSL}(2,7)$, $\mathrm{PGL}(2,7)$, M_9,

PGL(2, 9), SL(2, 8), PΓL(2, 8), PSL(2, 11), PSL(2, 13), PSL(2, 17), PSL(3, 4), M_{11}, Sz(8), $(A_5 \times C_3) \times_\lambda C_2$ *with* $A_5C_2 \simeq \Sigma_5$ *and* $C_3C_2 \simeq \Sigma_3$, M_{22}, PSL(3, 3), *and* PSL(2, 19).

Proof. We'll reason in a similar way as in Theorem 3.2 of [25].

If $S(G) = G$, then G is completely reducible, hence $G = G_1 \times \cdots \times G_s \times Z(G)$ with the G_i simple non-abelian groups. Therefore

$$5^s \cdot |Z(G)| - (s + |Z(G)| - 1) = r(G) - \beta(G) = j \leqq 11$$

and necessarily $s = 1$ and $|Z(G)| \leqq 2$. Thus either $G \in \{A_5 \times C_2, \mathrm{PSL}(2,7) \times C_2\}$ or G is a simple group with $r(G) \leqq 12$, hence from [1], G is isomorphic to one of the following groups: A_5, PSL(2, 7), A_6, PSL(2, 11), A_7, PSL(2, 13), SL(2, 8), PSL(3, 4), M_{11}, Sz(8), PSL(2, 17), M_{22}, PSL(3, 3), PSL(2, 19).

Now we can suppose $S(G) < G$, that is, $\alpha(G) \geqq 1$. Further, we deduce from Lemma 2.18 of [25] that $\alpha(G) \geqq 3$. If $\alpha(G) = 3$, then Lemma 2.20 of [25] implies that G is isomorphic to one of the following groups: M_9, Σ_5, $(A_5 \times C_3) \times_\lambda C_2$.

If $\alpha(G) = 4$, then it follows from Lemma 1.8 that $G \simeq \mathrm{PGL}(2,7)$. Suppose $\alpha(G) \geqq 5$. We have $3\beta(G) + \alpha(G) \leqq 11$ from Lemma 3.1 of [25], so $\beta(G) = 1$ or 2. If $\beta(G) = 2$, then $r(G) \leqq 11 + 2 = 13$. Let $L_1 \neq L_2$ be the minimal normal subgroups of G, then $S(G) = L_1 \times L_2$. If L_1 and L_2 are not solvable, then

$$r_G(S(G)) \geqq 1 + r_G(L_1^*) + r_G(L_2^*) + r_G(L_1^*) \cdot r_G(L_2^*) \geqq 1 + 3 + 3 + 3 \cdot 3 = 16,$$

but $\alpha(G) \geqq 5$ implies $r_G(S(G)) \leqq 13\text{–}5 = 8$, which is impossible. Thus, $L_1 \simeq C_p^t$ for some prime p and L_2 is non-solvable and isomorphic to $A \times \overset{e}{\cdots} \times A$ with A a non-abelian simple group. Reasoning as above, we now have $r_G(S(G)) \geqq 1 + 1 + 2 \cdot r_G(L_2^*)$. If $e \geqq 2$, then $A \times A$ has elements of orders 1, 2, p_1, p_2, $2p_1$, $2p_2$, $p_1\, p_2$, where $p_1 \neq p_2$ are two odd prime factors of $|A|$, thus $r_G(L_2^*) \geqq 7$, that is impossible. Therefore $e = 1$ and $L_2 = A$ is a simple group. We have $2(1 + r_G(L_2^*)) \leqq 13 - 5 = 8$, so $r_G(L_2^*) \leqq 3$ and $|\{o(g) \mid g \in L_2^*\}| \leqq 3$. Consequently $L_2 \simeq A_5$ by Lemma 2.12 of [25]. Besides, $L_1 \leqq C_G(L_2)$. Suppose $C_G(L_2) = L_1$, then $G/L_1 \leqq \mathrm{Aut}(L_2) = \Sigma_5$, hence $G/S(G) \simeq C_2$, and $r_G(L_1^*) \geqq (p^t - 1)/2$, therefore $(p^t - 1)/2 \leqq 1$, and necessarily $G = (C_3 \times A_5) \times_\lambda C_2$ being $\alpha(G) = 3$, impossible. Thus we can suppose $L_1 < C_G(L_2)$ and $G/S(G) \neq C_2$. By considering the different orders of elements in $S(G) = C_p^t \times A_5$, it follows that $r_G(S(G)) \geqq 8$ and $\alpha(G) \leqq 5$. Moreover, if $x \in C_G(L_2) - S(G)$, then every element of A_5 is centralized by x, so $xS(G)$ has elements of, at least, three different orders, hence $r_G(xS(G)) \geqq 3$ and consequently $r(G/S(G)) \leqq 4$ (otherwise, $\alpha(G) = r_G(xS(G)) + \Sigma_{i=1}^s r_G(x_iS(G))$ with $s \geqq 3$ implies $\alpha(G) \geqq 3 + 1 + 1 + 1 = 6$, impossible). If $r(G/S(G)) = 4$, then there exists $y \in G - S(G)$ such that

$r_G(yS(G))=1$, hence $|C_G(y)|\in\{2,3,4,5\}$ and necessarily $S(G)$ is solvable, impossible. If $G/S(G)=\langle\bar{x}\rangle\simeq C_3$, then $\alpha(G)=2\cdot r_G(xS(G))\geqq 6$, impossible. Finally, if $G/S(G)=\langle\bar{a}\rangle\times_f\langle\bar{b}\rangle\simeq\Sigma_3$, then $C_G(L_2)=S(G)\langle a\rangle$ and $r_G(bS(G))\leqq 2$, hence $|C_G(b)|=2$ or 4 and $S(G)$ is solvable. Thus $\beta(G)=1$ and $r(G)\leqq 12$. Set $S(G)=\underbrace{A\times\cdots\times A}_{e}$, with A a non-abelian simple group. As $\alpha(G)\geqq 5$, we have $r_G(S(G))\leqq 7$, hence $|\{o(g)\mid g\in S(G)\}|\leqq 7$ and this implies that $e\leqq 2$. If $e=2$ and $p_1\neq p_2$ are two odd prime numbers, divisors of $|A|$, then $S(G)$ has elements of order 1, 2, p_1, p_2, $2p_1$, $2p_2$, p_1p_2, hence $r_G(S(G))=7$ and $\alpha(G)=5$. Moreover, necessarily $|\{o(g)\mid g\in A^*\}|=3$, so $A\simeq A_5$. We have $C_G(S(G))=1$, because $\beta(G)=1$ and also

$$S(G)\lhd G\leqq \mathrm{Aut}(S(G))=\mathrm{Aut}(A_5)\sim\Sigma_2=\Sigma_5\sim\Sigma_2=(\Sigma_5\times\Sigma_5)\times_\lambda C_2,$$

being $\mathrm{Aut}(A_5\times A_5)\simeq C_2\sim C_2=D_8$. If $G/A_5^2\simeq C_2$, then $r(G)=2s+(25-s)/2$ and 2 divides $|A_5|^2$, so $s\geqq 2$, but $s\equiv 1 \pmod 2$, hence $s\geqq 3$ and $r(G)\geqq 6+11=17$, that is impossible. If $|G/S(G)|=4$ or $G/S(G)\simeq D_8$, then there exists $y\in G-S(G)$ such that $r_G(yS(G))=1$, hence $|C_G(y)|=4$ and $S(G)$ is solvable, impossible. Thus, necessarily $S(G)=A$ is a non-abelian simple group, $\beta(G)=1$, $C_G(A)=1$ and $A\lhd G\leqq\mathrm{Aut}(A)$. Further, $r(G)\leqq 12$ and $\alpha(G)\geqq 5$.

If $\alpha(G)\geqq 7$, then $r_G(S(G))\leqq 5$, hence $|\{o(g)\mid g\in A\}|\leqq 5$ and necessarily $A\in\{A_5,\mathrm{PSL}(2,7), A_6, \mathrm{SL}(2,8)\}$. We have $\mathrm{Aut}(A_5)\simeq\Sigma_5$, $\mathrm{Aut}(A_6)\simeq\mathrm{P\Gamma L}(2,9)$, $\mathrm{Aut}(\mathrm{PSL}(2,7))\simeq\mathrm{PGL}(2,7)$ and $\mathrm{Aut}(\mathrm{SL}(2,8))=\mathrm{P\Gamma L}(2,8)$, and the possible groups that appear here satisfy either $r(G)>12$ or $\alpha(G)<7$. Therefore $\alpha(G)\in\{5,6\}$ and consequently $r(G/S(G))\leqq 7$.

If $r(G/S(G))=7=\alpha(G)+1$, then $|C_G(x)|=|C_{\bar{G}}(\bar{x})|$ for each $x\in G-S(G)$ and Lemma 2.3 of [25] yields that $S(G)$ is abelian, impossible.

If $r(G/S(G))=5$ or 6, then, at least, there are $x,y\in G-S(G)$ such that $r_G(xS(G))=1=r_G(yS(G))$ and $\bar{x}$ does not conjugate with $\bar{y}$ in $\bar{G}$. Now, from an inspection of the tuples $\Delta_{\bar{G}}$ of the groups with 5 or 6 conjugate classes, we deduce from Lemma 2.13 of [25] that $S(G)$ is solvable, which is impossible. Thus we can suppose that $G/S(G)$ is isomorphic to one of the following groups: C_2, C_3, Σ_3, C_4, $C_2\times C_2$, D_{10} and A_4.

If $G/S(G)\simeq A_4$, we have $\alpha(G)=r_G(aS(G))+r_G(bS(G))+r_G(b^{-1}S(G))\leqq 6$ with $o(\bar{a})=2$ and $o(\bar{b})=3$, hence $r_G(bS(G))\leqq 2$, so $|C_G(b)|=3$ or 6 and $S(G)$ is solvable by Lemma 2.13 (cf. [25]). Similarly, the case $\bar{G}\simeq D_{10}$ cannot arise here.

Suppose $|\bar{G}|=4$, then there exists $b\in G-S(G)$ such that $r_G(bS(G))=2$, hence $\Delta_b=(8,8)$ and G has sectional rank at most 4. Now [8] and Lemmas 1.5 and 1.6 imply that there is not any group in this case.

TABLE 10

$G/S(G)$	G	$r(G)$
C_2	$C_{10}\times(H\times_f C_2)$	$r=15+5\lvert H\rvert$
	$C_5\times(H\times_\lambda C_4)=C_5\times(H\times_\lambda\langle b\rangle)$ with $h^b=h^{-1}\ \forall h\in H$	$r=15+5\lvert H\rvert$
C_3	$C_5\times(Y\times_f C_3)$	$r=5\cdot(3+(\lvert Y\rvert-1)/3)$
Σ_3	$C_2\times(C_2^4\times_\lambda\Sigma_3)=C_2\times(\langle x_1,y_1,x_2,y_2\rangle\times_\lambda\langle a,b\rangle)$ with $x_i^a=y_i,\ y_i^a=x_iy_i=y_i^b,\ x_i^b=x_i,\ i=1,2$	$r=20,\ \beta(G)=4$
	$C_2^4\times_\lambda DC_3=\langle x_1,y_1,x_2,y_2\rangle\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle)$ with $x_i^a=y_i,\ y_i^a=x_iy_i=y_i^b,\ x_i^b=x_i,\ i=1,2$	$r=20,\ \beta(G)=4$
	$C_7^2\times_\lambda(C_9\times_f C_2)=\langle x_1,x_2\rangle\times_\lambda(\langle a\rangle\times_f\langle b\rangle)$ with $x_1^a=x_2,\ x_2^a=x_1^{-1}x_2^{-1},\ x_1^b=x_1,\ x_2^b=x_1^{-1}x_2^{-1}$	$r=39,\ \beta(G)=2$
	$C_7^2\times_\lambda(C_3^2\times_f C_2)=\langle x_1,x_2\rangle\times_\lambda(\langle a_1,a_2\rangle\times_f\langle b\rangle)$ with $x_i^{a_1}=x_i,\ x_1^{a_2}=x_2,\ x_2^{a_2}=x_1^{-1}x_2^{-1},\ x_1^b=x_1,\ x_2^b=x_1^{-1}x_2^{-1}$	$r=39,\ \beta(G)=2$
	$C_5^3\times_\lambda\Sigma_3=\langle x_1,x_2,x_3\rangle\times_\lambda(\langle a\rangle\times_f\langle b\rangle)$ with $x_1^a=x_1,\ x_2^a=x_3,\ x_3^a=x_2^{-1}x_3^{-1},\ x_1^b=x_1^{-1},\ x_2^b=x_2,\ x_3^b=x_2^{-1}x_3^{-1}$	$r=35,\ \beta(G)=2$
	$C_2\times(C_2^2\times_\lambda(C_9\times_f C_2))=C_2\times(\langle x,y\rangle\times_\lambda(\langle a\rangle\times_f\langle b\rangle))$ with $x^a=y,\ y^a=xy,\ x^b=x,\ y^b=xy$	$r=18,\ \beta(G)=3$
	$C_2\times(C_2^2\times_\lambda(C_3^2\times_f C_2))=C_2\times(\langle x,y\rangle\times_\lambda(\langle a_1,a_2\rangle\times_f\langle b\rangle))$ with $x^{a_1}=x,\ y^{a_1}=y,\ x^{a_2}=y,\ y^{a_2}=xy,\ x^b=x,\ y^b=xy$	$r=18,\ \beta(G)=3$
	$(C_3\times C_9)\times_f C_2$	$r=15,\ \beta(G)=4$
C_4	$(C_{15}\times H)\times_\lambda C_4=(\langle x\rangle\times H)\times_\lambda\langle a\rangle$ with $x^a=x^{-1}$ and $H\langle a\rangle=H\times_f\langle a\rangle$	$r=18+15\cdot(\lvert H\rvert-1)/4$
	$C_2\times(C_5\times H)\times_\lambda C_4=C_2\times(\langle x\rangle\times H)\times_\lambda\langle a\rangle$ with $x^a=x^{-1}$ and $H\langle a\rangle=H\times_f\langle a\rangle$	$r=16+10(\lvert H\rvert-1)$
	$(C_5\times H)\times_\lambda C_8=(\langle x\rangle\times H)\times_\lambda\langle a\rangle$ with $x^a=x^{-1}$ $C_H(a^2)=1$ and $h^{a^4}=h\ \forall h\in H$	$r=16+10(\lvert H\rvert-1)/4$
C_2^2	$C_2\times(C_3\times_\lambda D_8)=C_2\times(\langle x\rangle\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle))$ with $x^a=x^{-1},\ x^b=x$	$r=18,\ \beta(G)=4$
	$C_2\times(C_3\times_\lambda D_8)=C_2\times(\langle x\rangle\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle))$ with $x^a=x,\ x^b=x^{-1}$	$r=18,\ \beta(G)=4$
	$C_2\times(C_3\times_\lambda Q_8)=C_2\times(\langle x\rangle\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle))$ with $x^a=x,\ x^b=x^{-1}$	$r=18,\ \beta(G)=4$
	$C_{12}\times_\lambda C_4=\langle a\rangle\times_\lambda\langle b\rangle$ with $a^b=a^{-1}$	$r=18,\ \beta(G)=4$
	$C_3\times_\lambda(C_4\times_\lambda C_4)=\langle x\rangle\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle)$ with $a^b=a^{-1},\ x^a=x^{-1},\ x^b=x$	$r=18,\ \beta(G)=4$
	$(C_3\times C_4\times C_2)\times_\lambda C_2=(\langle x\rangle\times\langle a\rangle\times\langle b\rangle)\times_\lambda\langle c\rangle$ with $a^b=a^{-1},\ x^a=x^{-1},\ x^b=x$	$r=18,\ \beta(G)=4$
	$C_3\times_\lambda((C_4\times C_2)\times_\lambda C_2)=\langle x\rangle\times_\lambda(((\langle a\rangle\times\langle b\rangle)\times_\lambda\langle c\rangle)$ with $a^c=ab,\ b^c=b,\ x^a=x^{-1},\ x^c=x,\ x^b=x$	$r=18,\ \beta(G)=4$
	$D_{26}\times\Sigma_3$	$r=24,\ \beta(G)=2$
	$D_{22}\times D_{10}$	$r=28,\ \beta(G)=2$
	$(C_3^2\times_f C_2)\times D_{14}$	$r=30,\ \beta(G)=2$
	$(C_3\times C_{11}\times C_3)\times_\lambda C_2^2=(\langle x_1\rangle\times\langle x_2\rangle\times\langle x_3\rangle)\times_\lambda\langle a_1,a_2\rangle$ with $x_1^{a_1}=x_1,\ x_2^{a_1}=x_2^{-1},\ x_3^{a_1}=x_3^{-1},\ x_1^{a_2}=x_1^{-1},\ x_2^{a_2}=x_2,\ x_3^{a_2}=x_3^{-1}$	$r=39,\ \beta(G)=3$
	$(C_3^3\times C_5)\times_\lambda C_2^2=(\langle x_1,x_2,x_3\rangle\times\langle x_4\rangle)\times_\lambda\langle a_1,a_2\rangle$ with $x_1^{a_1}=x_1,\ x_2^{a_1}=x_2^{-1},\ x_3^{a_1}=x_3^{-1},\ x_4^{a_1}=x_4^{-1}$, $x_1^{a_2}=x_1^{-1},\ x_2^{a_2}=x_2,\ x_3^{a_2}=x_3,\ x_4^{a_2}=x_4$	$r=48,\ \beta(G)=6$
	$C_3^2\times_\lambda(C_4\times C_2)=\langle x_1,x_2\rangle\times_\lambda(\langle a_1\rangle\times\langle a_2\rangle)$ with $x_1^{a_1}=x_1^{-1},\ x_2^{a_1}=x_2^{-1},\ x_1^{a_2}=x_1,\ x_2^{a_2}=x_2^{-1}$	$r=18,\ \beta(G)=3$
	$C_2\times\Sigma_3\times\Sigma_3$	$r=18,\ \beta(G)=3$
	$C_4\times D_{10}$	$r=16,\ \beta(G)=2$

TABLE 10 (contd.)

$G/S(G)$	G	$r(G)$
	$(C_3\times C_7^2)\times_\lambda C_2^2=(\langle x_1\rangle\times\langle x_2,x_3\rangle)\times_\lambda\langle a_1,a_2\rangle$ with $x_1^{a_1}=x_1,\ x_2^{a_1}=x_2^{-1},\ x_3^{a_1}=x_3^{-1},\ x_1^{a_2}=x_1^{-1},\ x_2^{a_2}=x_2,\ x_3^{a_2}=x_3^{-1}$	$r=51,\ \beta(G)=3$
	$(C_7\times C_5^2)\times_\lambda C_2^2=(\langle x_1\rangle\times\langle x_2,x_3\rangle)\times_\lambda\langle a_1,a_2\rangle$ with $x_1^{a_1}=x_1,\ x_2^{a_1}=x_2^{-1},\ x_3^{a_1}=x_3^{-1},\ x_1^{a_2}=x_1^{-1},\ x_2^{a_2}=x_2,\ x_3^{a_2}=x_3^{-1}$	$r=58,\ \beta(G)=3$
D_{10}	$(C_3\times C_2^4)\times_\lambda D_{10}=(\langle x\rangle\times\langle y_1,\dots,y_4\rangle)\times_\lambda(\langle a\rangle\times_f\langle b\rangle)$ with $x^a=x,\ x^b=x^{-1},\ y_1^a=y_2,\ y_2^a=y_3,\ y_3^a=y_4,\ y_4^a=y_1y_2y_3y_4,$ $y_1^b=y_1,\ y_2^b=y_1y_2y_3y_4,\ y_3^b=y_4,\ y_4^b=y_3$	$r=18,\ \beta(G)=2$
A_4	$C_2\times\mathrm{SL}(2,3)$	$r=14,\ \beta(G)=3$
C_5	—	—
Q_8	$(C_5\times H)\times_\lambda Q_8=(\langle x\rangle\times H)\times_\lambda\langle a,b\rangle$ with $x^a=x^{-1}$, $x^b=x$ and $HQ_8=H\times_f Q_8$	$r=13+5(\lvert H\rvert-1)/8$
D_8	$C_7^2\times_\lambda D_8=\langle x,y\rangle\times(\langle a\rangle\times_\lambda\langle b\rangle)$ with $x^a=y,\ y^a=x^{-1},\ x^b=x,\ y^b=y^{-1}$	$r=20,\ \beta(G)=1$
D_{14}	—	—
Hol C_5	—	—
$C_7\times_f C_3$	$C_2^6\times_\lambda(C_7\times_f C_3)=\langle x_1,\dots,x_6\rangle\times_\lambda(\langle a\rangle\times_f\langle b\rangle)$ with $x_i^a=x_{i+1},\ x_6^a=x_1\cdots x_6,\ x_1^b=x_1,\ x_2^b=x_3,$ $x_3^b=x_5,\ x_4^b=x_1\cdots x_6,\ x_5^b=x_2,\ x_6^b=x_4$	$r=16,\ \beta(G)=1$
	$C_2^6\times_\lambda(C_7\times_f C_3)=\langle x_1,\dots,x_6\rangle\times_\lambda(\langle a\rangle\times_f\langle b\rangle)$ with $x_1^a=x_2,\ x_2^a=x_3,\ x_3^a=x_1x_2,\ x_4^a=x_5,\ x_5^a=x_6,\ x_6^a=x_4x_5,$ $x_1^b=x_1,\ x_2^b=x_3,\ x_3^b=x_2x_3,\ x_4^b=x_4,\ x_5^b=x_6,\ x_6^b=x_5x_6$	$r=16,\ \beta(G)=3$
Σ_4	$C_3^3\times_\lambda\Sigma_4=\langle x_1,x_2,x_3\rangle\times_\lambda(\langle a_1,a_2\rangle\times_\lambda(\langle b\rangle\times_\lambda\langle c\rangle))$ with $a_1^b=a_2,\ a_2^b=a_1a_2,\ a_1^c=a_1,\ a_2^c=a_1a_2,\ b^c=b^{-1}$ $b^3=1,\ c^2=1,\ a_i^2=1,\ i=1,2,\ x_1^{a_1}=x_1,\ x_2^{a_1}=x_2^{-1},$ $x_3^{a_1}=x_3^{-1},\ x_1^{a_2}=x_1^{-1},\ x_2^{a_2}=x_2,\ x_3^{a_2}=x_3^{-1},\ x_1^b=x_2,$ $x_2^b=x_3,\ x_3^b=x_1,\ x_1^c=x_1^{-1},\ x_2^c=x_3^{-1},\ x_3^c=x_2^{-1}$	$r=14,\ \beta(G)=1$
A_5	$C_2^4\times_\lambda\mathrm{SL}(2,5)=\langle x_1,x_2,x_3,x_4\rangle\times_\lambda\langle a,b,c\rangle$ with $a^5=b^3=c^2=1,$ $(ab)^2=c,\ a^c=a,\ b^c=b,\ x_1^a=x_2,\ x_2^a=x_3,\ x_3^a=x_4,$ $x_4^a=x_1x_2x_3x_4,\ x_1^b=x_1x_2,\ x_2^b=x_1,\ x_3^b=x_2x_3x_4,\ x_4^b=x_1x_3$	$r=14,\ \beta(G)=2$
	$C_2^5\cdot A_5$ the only perfect extension of A_5 by C_2^5 which admits neither complement nor supplement $C_2^5\cdot A_5=\langle x_1,x_2,x_3,x_4,x_5,c,d\rangle$ with $c^3=1=d^5,\ \langle x_1,\dots,x_5\rangle\simeq C_2^5,$ $(cd)^2=x_1,\ x_1^c=x_2,\ x_2^c=x_3,\ x_3^c=x_5,\ x_4^c=x_1,\ x_5^c=x_4,$ $x_1^d=x_3,\ x_2^d=x_1,\ x_3^d=x_2,\ x_4^d=x_1x_2x_3x_4x_5,\ x_5^d=x_4$	$r=14,\ \beta(G)=1$
C_6	$(C_2^4\times H)\times_\lambda C_6=(\langle x_1,y_1,x_2,y_2\rangle\times H)\times_\lambda\langle a\rangle$ with $x_i^a=y_i,\ y_i^a=x_iy_i,\ i=1,2,\ H\langle a\rangle=H\times_f\langle a\rangle$	$r=16+8(\lvert H\rvert-1)/3$
	$(C_5\times C_2^2\times D)\times_\lambda C_6=(\langle x\rangle\times\langle y,z\rangle\times D)\times_\lambda\langle a\rangle$ with $D=1$ or H, $x^a=x^{-1},\ y^a=z,\ z^a=yz,\ D\langle a\rangle=D\times_f\langle a\rangle$	$r=14+(10\lvert D\rvert-4)/3$

TABLE 10 (contd.)

$G/S(G)$	G	$r(G)$
	$C_2\times(H\times_f C_6)$	$r=12+(\|H\|-1)/3$
	$H\times_\lambda C_{12}=H\times_\lambda\langle a\rangle$ with $C_H(a^6)=H$ and a, a^2, a^3 acting f.p.f. over H	$r=12+(\|H\|-1)/3$
D_{12}	$C_2^2\times_\lambda(C_3\times_\lambda D_8)=\langle x_1,x_2\rangle\times_\lambda(\langle c\rangle\times_\lambda\langle a,b\rangle)$ with $a^4=1=b^2$, $a^b=a^{-1}$, $c^b=c$, $c^a=c^{-1}$, $x_1^a=x_1$, $x_2^a=x_1x_2$, $x_1^c=x_2$, $x_2^c=x_1x_2$, $x_1^b=x_1$, $x_2^b=x_2$	$r=14$, $\beta(G)=2$
	$C_2^2\times_\lambda(C_3\times_\lambda D_8)=\langle x_1,x_2\rangle\times_\lambda(\langle c\rangle\times_\lambda\langle a,b\rangle)$ with $a^4=1=b^2$, $a^b=a^{-1}$, $c^b=c^{-1}$, $c^a=c$, $x_1^a=x_1$, $x_2^a=x_2$, $x_1^c=x_2$, $x_2^c=x_1x_2$, $x_1^b=x_1$, $x_2^b=x_1x_2$	$r=14$, $\beta(G)=2$
	$C_2^2\times_\lambda(C_3\times_\lambda Q_8)=\langle x_1,x_2\rangle\times_\lambda(\langle c\rangle\times_\lambda\langle a,b\rangle)$ with $c^a=c^{-1}$, $c^b=c$, $a^b=a^{-1}$, $x_1^a=x_1$, $x_2^a=x_1x_2$, $x_1^c=x_2$, $x_2^c=x_1x_2$, $x_1^b=x_1$, $x_2^b=x_2$	$r=14$, $\beta(G)=2$
DC_3	$C_2\times(H\times_f DC_3)$	$r=12+(\|H\|-1)/6$
	$H\times_\lambda(C_3\times_\lambda C_8)=H\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle)$ with $a^b=a^{-1}$, $C_H(b^2)=H$ and a, b^2 acting f.p.f. over H	$r=12+(\|H\|-1)/6$
	$H\times_\lambda(C_9\times_\lambda C_4)=H\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle)$ with $a^b=a^{-1}$, $C_H(a^3)=H$, and a, b^2 acting f.p.f. over H	$r=12+(\|H\|-1)/4$
	$(C_3\times H)\times_\lambda DC_3=(\langle x\rangle\times H)\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle)$ with $x^a=x$, $x^b=x^{-1}$, $H\langle a,b\rangle=H\times_f\langle a,b\rangle$	$r=12+(\|H\|-1)/4$
	$(C_5\times C_2^2\times D)\times_\lambda DC_3=(\langle x\rangle\times\langle x_1,x_2\rangle\times D)\times_\lambda(\langle a\rangle\times_\lambda\langle b\rangle)$ with $D=1$ or H, $x^a=x$, $x^b=x^2$, $x_1^a=x_2$, $x_2^a=x_1x_2$, $x_1^b=x_1$, $x_2^b=x_1x_2$ and $D\langle a,b\rangle=D\times_f\langle a,b\rangle$	$r=14+5(\|D\|-1)/3$
D_{18}	——	——
$C_3^2\times_f C_2$	$C_2^4\times_\lambda(C_3^2\times_f C_2)=\langle x_1,x_2,x_3,x_4\rangle\times_\lambda(\langle a_1,a_2\rangle\times_f\langle b\rangle)$ with $x_1^{a_1}=x_1$, $x_2^{a_1}=x_2$, $x_3^{a_1}=x_4$, $x_4^{a_1}=x_3x_4$, $x_1^b=x_1$, $x_2^b=x_1x_2$, $x_1^{a_2}=x_2$, $x_2^{a_2}=x_1x_2$, $x_3^{a_2}=x_3$, $x_4^{a_2}=x_4$, $x_3^b=x_3$, $x_4^b=x_3x_4$	$r=14$, $\beta(G)=2$
$C_3^2\times_f C_4$	$C_2^4\times_\lambda(C_3^2\times_f C_4)=\langle x_1,\dots,x_4\rangle\times_\lambda(\langle a_1,a_2\rangle\times_f\langle b\rangle)$ with $a_1^b=a_2$, $a_2^b=a_1^{-1}$, $x_1^{a_1}=x_2$, $x_2^{a_1}=x_1x_2$, $x_3^{a_1}=x_3$, $x_4^{a_1}=x_4$, $x_1^{a_2}=x_1$, $x_2^{a_2}=x_2$, $x_3^{a_2}=x_4$, $x_4^{a_2}=x_3x_4$, $x_1^b=x_3$, $x_2^b=x_4$, $x_3^b=x_1$, $x_4^b=x_1x_2$	$r=13$, $\beta(G)=1$
$C_3^2\times_f Q_8$	——	——
PSL(2,7)	——	——
C_7	——	——
Q_{16}	$(C_3\times H)\times_\lambda Q_{16}=(\langle x\rangle\times H)\times_\lambda\langle a,b\rangle$ with $x^a=x^{-1}$, $x^b=x$, and $HQ_{16}=H\times_f Q_{16}$	$r=12+3\cdot(\|H\|-1)/16$
SD_{16}	$C_3^4\times_{\lambda_1}SD_{16}=\langle x_1,y_1,x_2,y_2\rangle\times_\lambda\langle a,b\rangle$ with $x_i^a=y_i$, $y_i^a=x_iy_i$, $x_i^b=x_i$, $y_i^b=x_iy_i^{-1}$, $i=1,2$, $a^8=1=b^2$, $a^b=a^3$	$r=18$, $\beta(G)=4$
	$C_3^4\times_{\lambda_2}SD_{16}=\langle x_1,x_2,x_3,x_4\rangle\times_\lambda\langle a,b\rangle$ with $x_1^a=x_2$, $x_2^a=x_3$, $x_3^a=x_4$, $x_4^a=x_1^{-1}$, $x_1^b=x_1$, $x_2^b=x_4$, $x_3^b=x_3^{-1}$, $x_4^b=x_2$	$r=18$, $\beta(G)=1$

TABLE 10 (contd.)

$G/S(G)$	G	$r(G)$
$SL(2,3)$	$(C_2^2 \times H) \times_\lambda SL(2,3) = ((\langle x_1, x_2 \rangle \times H) \times_\lambda (\langle a, b \rangle \times_\lambda \langle c \rangle))$ with $a^c = b$, $b^c = ab$, $x_i^a = x_i = x_i^b$, $i = 1,2$, $x_1^c = x_2$, $x_2^c = x_1x_2$, and $H \cdot SL(2,3) = H \times_f SL(2,3)$	$r = 12 + (\lvert H \rvert - 1)/6$
$C_{13} \times_f C_3$	$C_3^3 \times_\lambda (C_{13} \times_f C_3) = \langle x_1, x_2, x_3 \rangle \times_\lambda \langle a, b \rangle$ with $x_1^a = x_2$, $x_2^a = x_3$, $x_3^a = x_1x_2 = x_2^b$, $x_1^b = x_1$, $x_3^b = x_1x_2^{-1}x_3$	$r = 13$, $\beta(G) = 1$

There are no groups G such that $G/S(G)$ is isomorphic to one of the following list: D_{16}, D_{22}, $C_7 \times_f C_6$, $C_{13} \times_f C_4$, $C_{11} \times_f C_5$.

A_6	$C_2^4 \times_\lambda A_6 = \langle x_1, x_2, x_3, x_4 \rangle \times_\lambda \langle a, b \rangle$ with $a^5 = b^5 = 1$, $(ab)^2 = (a^{-1}b)^4 = 1$, $x_1^a = x_4$, $x_2^a = x_1x_4$, $x_3^a = x_2x_4$, $x_4^a = x_3x_4$, $x_1^b = x_1x_2$, $x_2^b = x_2x_3$, $x_3^b = x_2x_4$, $x_4^b = x_1x_3x_4$	$r = 12$, $\beta(G) = 1$
Σ_5	$\Sigma_5^{(1)}$	$r = 12$, $\beta(G) = 1$
	$\Sigma_5^{(2)}$	$r = 12$, $\beta(G) = 1$
	$C_2^4 \times_\lambda \Sigma_5 = \langle x_1, x_2, x_3, x_4 \rangle \times_\lambda \langle a, b, d \rangle$ with $a^5 = b^3 = (ba)^2 = 1 = d^2$, $(a^2ba^2)^d a^3 = 1$, $x_1^a = x_2$, $x_2^a = x_3$, $x_3^a = x_4$, $x_4^a = x_1x_2x_4$, $x_1^b = x_1x_2$, $x_2^b = x_1$, $x_3^b = x_2x_3x_4$, $x_4^b = x_1x_3$, $x_1^d = x_1$, $x_2^d = x_2$, $x_3^d = x_2x_3$, $x_4^d = x_1x_2x_4$	$r = 12$, $\beta(G) = 1$
C_2^3	—	—
C_8	$(C_{13} \times H) \times_\lambda C_8 = ((\langle x \rangle \times H) \times_\lambda \langle a \rangle$ with $x^a = x^5$, and $H\langle a \rangle = H \times_f \langle a \rangle$	$r = 14 + 13 \cdot (\lvert H \rvert - 1)/8$
	$(C_3 \times H) \times_\lambda C_8 = ((\langle x \rangle \times H) \times_\lambda \langle a \rangle$ with $x^a = x^{-1}$, and $H\langle a \rangle = H \times_f \langle a \rangle$	$r = 12 + 3 \cdot (\lvert H \rvert - 1)/8$
$C_4 \times C_2$	$C_5 \times_\lambda M_{16} = \langle x \rangle \times_\lambda (\langle a \rangle \times_\lambda \langle b \rangle)$ with $a^8 = 1 = b^2$, $x^a = x^2$, $x^b = x$	$r = 14$, $\beta(G) = 2$
	$C_5 \times_\lambda M_{16} = \langle x \rangle \times_\lambda (\langle a \rangle \times_\lambda \langle b \rangle)$ with $x^a = x^2$, $x^b = x^{-1}$,	$r = 14$, $\beta(G) = 2$
	$C_5 \times_\lambda (C_4 \times_\lambda C_4) = \langle x \rangle \times_\lambda (\langle a \rangle \times_\lambda \langle b \rangle)$ with $a^b = a^{-1}$, $x^a = x$, $x^b = x^2$	$r = 14$, $\beta(G) = 2$
	$C_5 \times_\lambda ((C_4 \times C_2) \times_\lambda C_2) = \langle x \rangle \times_\lambda (((\langle a \rangle \times \langle b \rangle) \times_\lambda \langle c \rangle)$ with $a^c = ab$, $b^c = b$, $x^a = x^2$, $x^c = x$, $x^b = x$	$r = 14$, $\beta(G) = 2$
	$(C_5 \times C_3^2) \times_\lambda (C_4 \times C_2) = ((\langle x \rangle \times \langle y_1, y_2 \rangle) \times_\lambda (\langle a \rangle \times \langle b \rangle)$ with $x^a = x^2$, $x^b = x$, $y_1^a = y_2$, $y_2^a = y_1^{-1}$, $y_1^b = y_1^{-1}$, $y_2^b = y_2^{-1}$	$r = 18$, $\beta(G) = 2$

If $r(G/S(G)) = 8$ and $\lvert G/S(G) \rvert > 8$, then $G/S(G) = C_2^4 \times_f C_5$ and we have:

$C_2^4 \times_f C_5$	$P \times_f C_5$ with P satisfying $P/C_2^4 \simeq C_2^4$, $Z(P) = P' \simeq C_2^4$	$r = 14$, $\beta(G) = 1$

If $r(G/S(G)) = 9$, then $G/S(G) \simeq C_9$ and we have:

C_9	$(C_2^2 \times Y) \times_\lambda C_9 = ((\langle x, y \rangle \times Y) \times_\lambda \langle a \rangle$ with $x^a = y$, $y^a = xy$, $Y\langle a \rangle = Y \times_f \langle a \rangle$	$r = 12 + 4 \cdot (\lvert Y \rvert - 1)/9$

If $r(G/S(G)) = 10$, then $G/S(G) \in \{M_{16}, C_2^4 \times_\lambda \Sigma_3, C_2^4 \times_\lambda C_6\}$ and we have:

M_{16}	$C_5^2 \times_\lambda M_{16} = \langle x_1, x_2 \rangle \times_\lambda \langle a, b \rangle$ with $a^8 = 1 = b^2$, $a^b = a^5$, $x_1^a = x_2$, $x_2^a = x_1^2$, $x_1^b = x_1$, $x_2^b = x_2^{-1}$	$r = 13$, $\beta(G) = 1$
$C_2^4 \times_\lambda \Sigma_3$	$P_1 \times_\lambda \Sigma_3 = P_1 \times_\lambda \langle a, b \rangle$ with $a^3 = b^2 = 1$, $a^b = a^{-1}$, $P_1 = C_2^4 \times_\lambda C_2^2 = \langle z_1, z_2, a_1, a_2 \rangle \times_\lambda \langle b_1, b_2 \rangle$ with	$r = 12$, $\beta(G) = 1$

TABLE 10 (contd.)

$G/S(G)$	G	$r(G)$
	$z_i^{b_j} = z_i,\ i,j = 1,2,\ a_1^{b_1} = a_1z_1,\ a_1^{b_2} = a_1z_2,\ a_2^{b_1} = a_2z_2,$	
	$a_2^{b_2} = a_2z_1z_2$ and $b_1^a = b_1b_2,\ b_2^a = b_1,\ b_1^b = b_2,\ b_2^b = b_1,$	
	$z_1^a = z_2,\ z_2^a = z_1z_2,\ a_1^a = a_1a_2,\ a_2^a = a_1,\ z_1^b = z_1z_2,\ z_2^b = z_2,$	
	$a_1^b = a_2,\ a_2^b = a_1$ (P_1 is a 2-group of type PSL(3, 4))	
$C_2^4 \times_\lambda C_6$	$P_1 \times_\lambda C_6 = P_1 \times_\lambda \langle ab \rangle$ with P_1 as above, and	$r = 12,\ \beta(G) = 1$
	$z_1^a = z_2,\ z_2^a = z_1z_2,\ a_1^a = a_1a_2,\ a_2^a = a_1,\ b_1^a = b_1b_2,\ b_2^a = b_1,$	
	$z_1^b = z_1,\ z_2^b = z_2,\ a_1^b = b_1b_2,\ a_2^b = b_1,\ b_1^b = a_2,\ b_2^b = a_1a_2$	
	$P_2 \times_\lambda C_6 = P_2 \times_\lambda \langle \alpha\beta \rangle$ with $P_2 = C_4^2 \cdot C_4^2 = \langle a, b \rangle \cdot \langle c, d \rangle,$	$r = 12,\ \beta(G) = 1$
	$[a, b] = [c, d] = 1,\ c^2 = b^2,\ d^2 = a^2b^2,\ [a, c] = a^2,$	
	$[a, d] = [b, c] = d^2,\ [b, d] = b^2$ and $C_6 = \langle \alpha\beta \rangle$ with relations	
	$a^\alpha = b,\ b^\alpha = a^{-1}b^{-1},\ c^\alpha = d,\ d^\alpha = c^{-1}d^{-1},$	
	$a^\beta = cd,\ b^\beta = c^{-1},\ c^\beta = b^{-1},\ d^\beta = ab$	
	(P_2 is a 2-group of type PSU(3, 4))	

If $\bar{G} = \langle \bar{a} \rangle \times_f \langle \bar{b} \rangle \simeq \Sigma_3$, then Lemmas 2.4 and 2.13 of [25] yield $r_G(aS(G)) \geqq 3$ and $r_G(bS(G)) \geqq 4$ respectively, impossible.

So then, either $G/S(G) \simeq C_3$ or $G/S(G) \simeq C_2$ with $\alpha(G) \in \{5, 6\}$ and $r(G) \leqq 12$.

If $G/S(G) = \langle \bar{b} \rangle \simeq C_3$, then necessarily $r_G(bS(G)) = 3 = r_G(b^{-1}S(G))$, hence $\alpha(G) = 6$ and $r_G(A) = 6$. If $|\{o(g) \mid g \in A\}| \leqq 5$, then A is isomorphic to one of the following groups: A_5, A_6, PSL(2, 7), SL(2, 8), so $G = \mathrm{P\Gamma L}(2, 8)$ $(\alpha(G) = 6)$. On the other hand, if $|\{o(g) \mid g \in A\}| = 6$, then $r(G) = 11$ or 12 and $r_G(A) = 6$ implies that "$a_1 \sim_G a_2$ iff $o(a_1) = o(a_2)$" for every $a_1, a_2 \in A$. Let s be the number of conjugate classes of A fixed by conjugation of b. Then $6 = \alpha(G) = s \cdot 2$ implies $s = 3$ and

$$r(A) = 3 + (r_G(A) - 3) \cdot 3 = 12,$$

hence $A \in \{M_{22}, \mathrm{PSL}(3, 3), \mathrm{PSL}(2, 19)\}$ which is impossible.

Finally, we consider only the case $G/S(G) \simeq C_2$. Then $r(G) = 2s + (r(A) - s)/2$ with $s = \alpha(G)$, and $r(A) = s + (r_G(A) - s) \cdot 2 = s + (6 - s) \cdot 2$. If $s = 6$, then $r(A) = 6$ and $A = \mathrm{PSL}(2, 7)$, impossible. Thus we have $s = 5$ and $r(A) = 7$, hence either $G \simeq \mathrm{PSL}(2, 9)$ $(\alpha(G) = 5)$ or $G \simeq \Sigma_6$ $(\alpha(G) = 5)$.

LEMMA 1.14. *Let G be a non-nilpotent group with $S(G)$ abelian and satisfying the conditions $\alpha(G) = 10$ and $r(G/S(G)) \leqq 10$. Then G is isomorphic to one group of Table* 10.

PROOF. The reasonings are similar to the ones followed in Lemma 4.2 of [25] for $\alpha(G) \leqq 9$, and for that reason we don't repeat them here.

LEMMA 1.15. *Let G be a non-nilpotent group with $S(G)$ solvable. If $\alpha(G)=10$ and $r(G/S(G))=11$, then G is isomorphic to one of the following groups*:

(1) $H\times_f C_{11}$ $(r=11+(|H|-1)/11)$,

(2) $Y\times_f Q_2$ $(r=11+(|Y|-1)/27)$,

(3) $H\times_f Q_{32}$ $(r=11+(|H|-1)/32)$.

PROOF. Let's assume $r(G/S(G))=11$. Then $|C_{\bar{G}}(\bar{x})|=|C_G(x)|$ for every $x\in G-S(G)$, where $\bar{G}=G/S(G)$, and the result follows immediately from Lemma 2.3 (cf. [25]) observing the tuples $\Delta_{\bar{G}}$ for $r(\bar{G})=11$ from Table 3 of [25].

LEMMA 1.16. *Let G be a nilpotent group such that $\alpha(G)\leqq 10$. Then G is isomorphic to one of the following groups*:

Abelian: 1, C_4, C_8, $C_2\times C_4$, C_9, $C_4\times C_2^2$, C_{12}, C_{20}, *and* $Y=C_2^e\times C_{p_1}^{t_1}\times\cdots\times C_{p_s}^{t_s}$.

Non-abelian: D_8, Q_8, Q_1, Q_2, $C_3\times D_8$, $C_3\times Q_8$, $C_2\times D_8$, $C_2\times Q_8$, $C_4\times_\lambda C_4=\langle a\rangle\times_\lambda\langle b\rangle$ *with* $a^b=a^{-1}$, $(C_4\times C_2)\times_{\lambda_1}C_2=(\langle a\rangle\times\langle b\rangle)\times_{\lambda_1}\langle c\rangle$ *with* $a^c=ab$, $b^c=b$, $(C_4\times C_2)\times_{\lambda_2}C_2=(\langle a\rangle\times\langle b\rangle)\times_{\lambda_2}\langle c\rangle$ *with* $a^c=a$, $b^c=a^2b$, D_{16}, SD_{16}, Q_{16}, $D_{16}\times C_2$, $\mathrm{SD}_{16}\times C_2$, $Q_{16}\times C_2$, $(C_8\times C_2)\times_\lambda C_2=(\langle a\rangle\times\langle b\rangle)\times_\lambda\langle c\rangle$ *with* $a^c=a^{-1}b$, $b^c=b$, M_{16}, $(C_8\times C_2)\cdot C_4=(\langle a\rangle\times\langle b\rangle)\cdot\langle c\rangle$ *with* $c^2=a^4$, $[b,c]=1$, $a^c=a^{-1}b$, $C_8\times_{\lambda_1}C_4=\langle a\rangle\times_{\lambda_1}\langle b\rangle$ *with* $a^b=a^{-1}$, $C_8\times_{\lambda_2}C_4=\langle a\rangle\times_{\lambda_2}\langle b\rangle$ *with* $a^b=a^3$, $C_2^4\times_\lambda C_2=\langle a_1,a_2,a_3,a_4\rangle\times_\lambda\langle b\rangle$ *with* $a_1^b=a_1$, $a_2^b=a_2$, $a_3^b=a_1a_3$, $a_4^b=a_2a_4$, $C_4^2\times_{\lambda_1}C_2=(\langle a\rangle\times\langle b\rangle)\times_{\lambda_1}\langle c\rangle$ *with* $a^c=a^{-1}$, $b^c=b^{-1}$, $(C_4\times C_4)\underset{1}{\cdot}C_4=(\langle a\rangle\times\langle b\rangle)_1\cdot\langle c\rangle$ *with* $c^2=a^2$, $a^c=a^{-1}$, $b^c=b^{-1}$, $(C_2^2\times C_4)\times_{\lambda_1}C_2=(\langle a_1,a_2\rangle\times\langle a_3\rangle)\times_{\lambda_1}\langle b\rangle$ *with* $a_1^b=a_1$, $a_2^b=a_1a_2$, $a_3^b=a_3^{-1}$, $(C_2^2\times C_4)\cdot C_4=(\langle a_1,a_2\rangle\times\langle a_3\rangle)\cdot\langle b\rangle$ *with* $a_1^b=a_1$, $a_2^b=a_1a_2$, $a_3^b=a_3^{-1}$, $b^2=a_3^2$, $(C_2^2\times C_4)\times_{\lambda_2}C_2=(\langle a_1,a_2\rangle\times\langle a_3\rangle)\times_{\lambda_2}\langle b\rangle$ *with* $a_1^b=a_1$, $a_2^b=a_1a_2$, $a_3^b=a_1^2a_3$, $C_4^2\times_{\lambda_2}C_2=(\langle a\rangle\times\langle b\rangle)\times_{\lambda_2}\langle c\rangle$ *with* $a^c=a^{-1}$, $b^c=a^2b^{-1}$, $(C_4\times C_4)_2\cdot C_4=(\langle a\rangle\times\langle b\rangle)_2\cdot\langle c\rangle$ *with* $a^c=a^{-1}$, $b^c=a^2b^{-1}$, $c^2=(ab)^2$, $\mathrm{Hol}\,C_8$, D_{32}, SD_{32}, Q_{32}, $(C_8\times_\lambda C_2)\cdot C_4=(\langle a\rangle\times_\lambda\langle b\rangle)\cdot\langle c\rangle$ *with* $a^b=a^5$, $a^c=ba$, $b^c=b$, $c^2=a^4$, $(C_8\times_\lambda C_2)\times_\lambda C_2=(\langle a\rangle\times_\lambda\langle b\rangle)\times_\lambda\langle c\rangle$ *with* $a^b=a^5$, $a^c=ba$, $b^c=b$, $C_2^3\times_\lambda C_4=\langle a,b,c\rangle\times_\lambda\langle d\rangle$ *with relations* $a^d=a$, $b^d=ab$, $c^d=abc$.

PROOF. If G is abelian, it is immediate. On the other hand, in case G is non-abelian, set $G=P_1\times\cdots\times P_t$ with the P_i Sylow p_i-subgroups of G. Then we have $S(G)=\Omega_1(Z(P_1))\times\cdots\times\Omega_1(Z(P_t))$. If $|G|$ is divisible by at least two prime numbers, it follows easily that $G\simeq C_3\times D_8$ of $G\simeq C_3\times Q_8$. So we can suppose that G is a p-group. If $p\neq 2$, then necessarily $p=3$ and $G\simeq Q_1$ or $G\simeq Q_2$. Suppose that G is a 2-group. We have $r(G/S(G))\leqq\alpha(G)+1=11$, hence $\bar{G}=G/S(G)$ is isomorphic to one of the following groups: C_2, C_4, C_2^2, D_8,

Q_8, SD_{16}, Q_{16}, D_{16}, C_8, $C_2\times C_4$, C_2^3, $C_2\times D_8$, $C_2\times Q_8$, $C_4\times_\lambda C_4$, $(C_4\times C_2)\times_{\lambda_1} C_2$, $C_8\times_\lambda C_2$, $(C_4\times C_2)\times_{\lambda_2} C_2$, D_{32}, Q_{32}, SD_{32}, $2^5\Gamma_6 a_1$, $2^5\Gamma_6 a_2$, $2^5\Gamma_7 a_1$, $2^5\Gamma_7 a_2$, $2^5\Gamma_7 a_3$.

If $G/S(G)$ is a cyclic group, then G is abelian, because $S(G)\leqq Z(G)$, which is impossible.

If $G/S(G)\simeq C_2^2$, then $\alpha(G)=\Sigma_{i=1}^3 r_G(d_iS(G))$, so there exists i such that $r_G(d_iS(G))\leqq 3$, consequently $3\geqq 2\cdot|S(G)|/4$, hence $|S(G)|\leqq 6$, and either $|S(G)|=2$, hence $G\in\{D_8, Q_8\}$, or $|S(G)|=4$ and G is isomorphic to one of the following groups: $C_2\times D_8$, $C_2\times Q_8$, $C_4\times_\lambda C_4$, $(C_4\times C_2)\times_{\lambda_1} C_2$.

If $G/S(G)\simeq D_8$, then

$$\alpha(G)=|S(G)|+r_G(a^2S(G))+r_G(bS(G))+r_G(abS(G)) \text{ and } |S(G)|\in\{2,4\}.$$

If $|S(G)|=2$, then G is isomorphic to one of the following groups: D_{16}, SD_{16}, Q_{16}, M_{16}, $(C_4\times C_2)\times_{\lambda_2} C_2$. If $|S(G)|=4$, then $G/S(G)\simeq D_8$ with $S(G)=\Omega_1(Z(G))\simeq C_2^2$. Besides, there exists $b\in G-S(G)$ such that $|C_G(b)|=8$, because $\alpha(G)\leqq 10$, so $Z(G)=S(G)$ and $r(G)\leqq 10+4=14$. Therefore $|G/G'|=8$ and consequently G is one of the ten groups of the first branch of the family Γ_3 (the second branch satisfies $|G/G'|=2^4$).

Suppose $G/S(G)\simeq Q_8$, then $\alpha(G)=3|S(G)|+r_G(a^2S(G))$, so $|S(G)|=2$, impossible.

Suppose $G/S(G)\in\{D_{16},\mathrm{SD}_{16},Q_{16}\}$ and let $\bar a$ be an element of order 8 in $G/S(G)$, then $2|C_G(a)\cap S(G)|=2\cdot|S(G)|\leqq 10-4$, so $|S(G)|=2$ and $r(G)\leqq 12$. Thus $G\in\{D_{32},\mathrm{SD}_{32},Q_{32}\}$.

In other cases we have $|S(G)|=4$ for $|G/S(G)|\leqq 16$ and $|S(G)|=2$ if $|G/S(G)|=32$, as follows from a simple inspection of the tuples $\Delta_{\bar G}$ and of the fact that $\alpha(G)\leqq 10$. Therefore $r(G)\leqq 14$, $|G/G'|\leqq 2^3$ and in these cases G is a stem group. Further, either G has order 32 and is in one of the families Γ_i, $i=2,3,4,6,7$, or G is a stem group of order 64 of the families Γ_{22} or Γ_{23}, being for these groups $r(G)=13$, $Z(G)=S(G)\simeq C_2$ and $\alpha(G)=11$, impossible.

Theorem 1.17. *$G\in\Phi_{11}$ if and only if G is one of the following groups*: M_{22}, $\mathrm{PSL}(3,3)$, $\mathrm{PSL}(2,19)$, $C_{37}\times_f C_6$, $C_3^4\times_f Q_{16}$, $C_{11}^2\times_f \mathrm{SL}(2,3)$, $C_2^4\times_{\lambda_2} A_5$, $C_3^4\times_f(C_5\times_\lambda C_4)$, $C_{19}^2\times_f \mathrm{SL}(2,5)$, $C_2^4\times_\lambda A_6$, $\Sigma_5^{(1)}$, $\Sigma_5^{(2)}$, $C_2^4\times_\lambda \Sigma_5$, $P_1\times_\lambda \Sigma_3$, $P_1\times_\lambda C_6$, $P_2\times_\lambda C_6$, $C_3^4\times_f(C_5\times_\lambda C_8)$, $C_5^2\times_\lambda C_4$, $C_2^2\times_\lambda(C_{15}\times_f C_2)$, $C_5\times_{\lambda_1} D_8$, $C_5\times_{\lambda_2} D_8$, $C_5\times_\lambda Q_8$, $C_2\times \mathrm{SL}(2,3)\}$ $\cup$ $2^5\Gamma_4$ $\cup$ $\{2^5\Gamma_3 a_i\,|\,1\leqq i\leqq 3\}$ $\cup$ $\{2^5\Gamma_3 c_i\,|\,1\leqq i\leqq 2\}$ $\cup$ $\{2^5\Gamma_3 d_1, 2^5\Gamma_3 d_2\}\cup\{C_3^4\times_f Q_8$, $(C_3\times C_9)\times_f C_2$, $C_7^2\times_f C_3\}$.

Proof. If is an immediate consequence from Theorem 2.17 [25], Lemma 2.18 [25], Lemma 2.19 [25], Lemma 2.20 [25], Theorem 3.2 [25], Lemma 4.1 [25], Lemma 4.2 [25], Lemma 4.5 [25], Lemma 4.8 [25], Lemma 4.11 [25], Lemma 4.14 [25], and Lemmas 1.8, 1.13, 1.14, 1.15 and 1.16.

Corollary 1.18. *$r(G)=12$ iff G is isomorphic to one of the groups listed in Table* 1.

Corollary 1.19. (1) *$r(G)=13$ and $\beta(G)>1$ iff G is isomorphic to one of the groups listed in Table* 2(i).

(2) *$r(G)=13$, $\beta(G)=1$ and $0\leqq\alpha(G)\leqq 4$ iff G is isomorphic to one of the group listed in Table* 2(ii).

(3) *$r(G)=13$, $\beta(G)=1$, $5\leqq\alpha(G)\leqq 10$ and $S(G)$ is solvable iff G is isomorphic to one of the group listed in Table* 2(iii).

. . . .

Corollary 1.26. (1) *There are no groups satisfying $r(G)=20$ and $\beta(G)>8$.*

(2) *$r(G)=20$, $\beta(G)\leqq 8$ and $0\leqq\alpha(G)\leqq 4$ iff G is isomorphic to one of the groups listed in Table* 9(ii).

(3) *$r(G)=20$, $\beta(G)\leqq 8$, $5\leqq\alpha(G)\leqq 1$ and $S(G)$ is solvable iff G is isomorphic to one of the group listed in Table* 9(iii).

Corollary 1.27. *Set $n\in\mathbf{N}$, $n\geqq 21$. Then $r(G)=n$ and $\beta(G)=n-a$ with $1\leqq a\leqq 11$, if and only if $G\in\{F'_{t_1,1}, F'_{t_2,2}, F'_{t_3,3}, F'_{t_4,4}, F'_{t_5,5}, F'_{t_6,6}, F'_{t_7,7}, F'_{t_8,8}\}$ with $t_1=\log_2 n$, $t_2=\log_3(2n-3)$, $t_3=(\log_2(3n-8))/2$, $t_4=\log_5(4n-15)$, $t_5=\log_7(6n-35)$, $t_6=(\log_2(7n-48))/3$, $t_7=(\log_3(8n-63))/2$, $t_8=\log_{11}(10n-99)$, and where $F'_{t,i}$ denote $F_{t,i}$ if t is a natural number, and is otherwise dropped from the list.*

Proof. It follows from Theorem 4.3 [25], Theorem 4.6 [25], Theorem 4.9 [25], Theorem 4.12 [25], Theorem 4.15 [25] and Theorem 1.17.

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